Lecture 5

Scientific Methodology

What is a scientific approach?

This question can be answered in a lot of different ways. We will try to do it by describing three somewhat different areas where we use science.

- Scientific attitude in every-day situations.
- Scientific methods in smaller research projects.
- Science in big scientific theories.

The scientific attitude

We can characterize the scientific method by the attitudes of scientists. According to Merton the following should be the attitudes. It is five principles gathered under the acronym CUDOS:

- Communalism - knowledge should be accessible for all people.
- Universalism - everyone should have the right to contribute.
- Disinterestedness - science should be objective and not ruled by special interests.
- Originality - the results should be new.
- Skepticism - scientists should be open to criticism.

Science in every-day situations

What does it mean to have a scientific attitude to things? Some suggestions:

- You are objective. Especially, you base your judgements on observations and verified facts.
- You realize to what extent you and everyone else can be biased by your/their perspective.
- You are curious and want to know facts.
- You have some knowledge of scientific methodology and try to apply it.

What scientific methodology?

Here are some scientific methods that also can be used in "simpler" situations:

- The HD-method for finding hypothesis. Use the formula H & A => E. (Lecture 2)
- Maximum Likelihood. Try to find H such that P(E I H) is maximal.
- If you are more advanced: Use Baye's formula for computing P(H I E). (Lecture 3)
- Realize that is A and B are correlated it doesn't have to mean that A is the cause of B. It can be the other way around, or neither. (Lecture 4)
- Use deduction.

Science in research projects

We identify three types of research projects:

- Exploratory research
- Testing-out research
- Problem-solving research

Exploratory research

- This is research on a new problem about which little is known.

- The problem may come from any part of the discipline; it may be a theoretical research puzzle or have an empirical basis.

- The research work will need to examine what theories and concepts are appropriate, developing new ones if necessary, and whether existing methodologies can be used.

- It obviously involves pushing out the frontiers of knowledge in the hope that something useful will be discovered.

Testing-out research

- In this type of research we are trying to find the limits of a previously proposed generalization.

- This is often termed the ‘null hypothesis', which we are bringing evidence to ‘overthrow’ - i.e. to show is inadequate.

- We can try to answer questions like: Does the theory apply at high temperatures? In new technology industries? With working-class parents? Before universal franchise was introduced?

- In this way we are able to make an original contribution and improve (by specifying, modifying, clarifying) the important generalizations in our discipline.

Problem-solving research

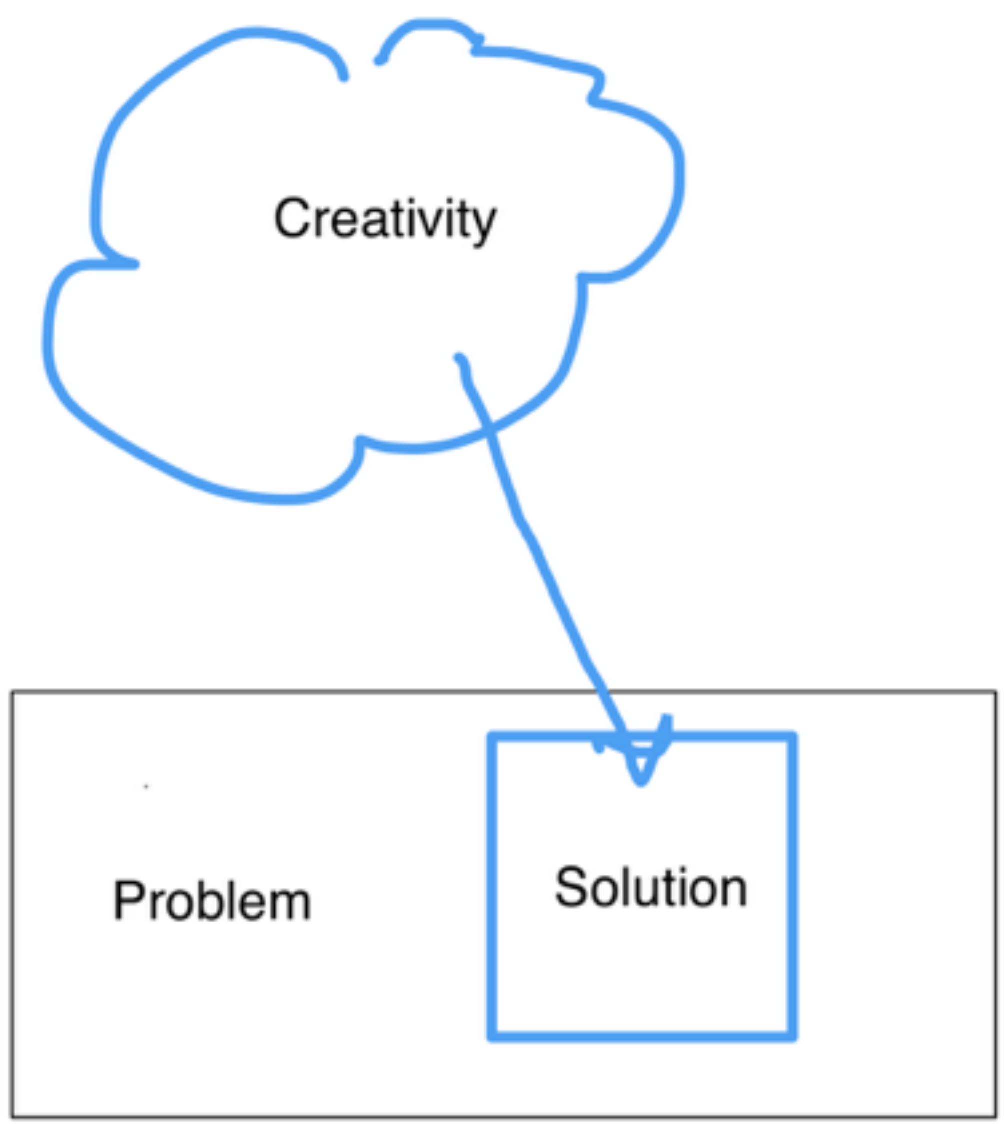
- In this type of research, we start from a particular problem in the real world, and bring together all the intellectual resources that can be brought to bear on its solution.

- The problem has to be defined and the method of solution has to be discovered.

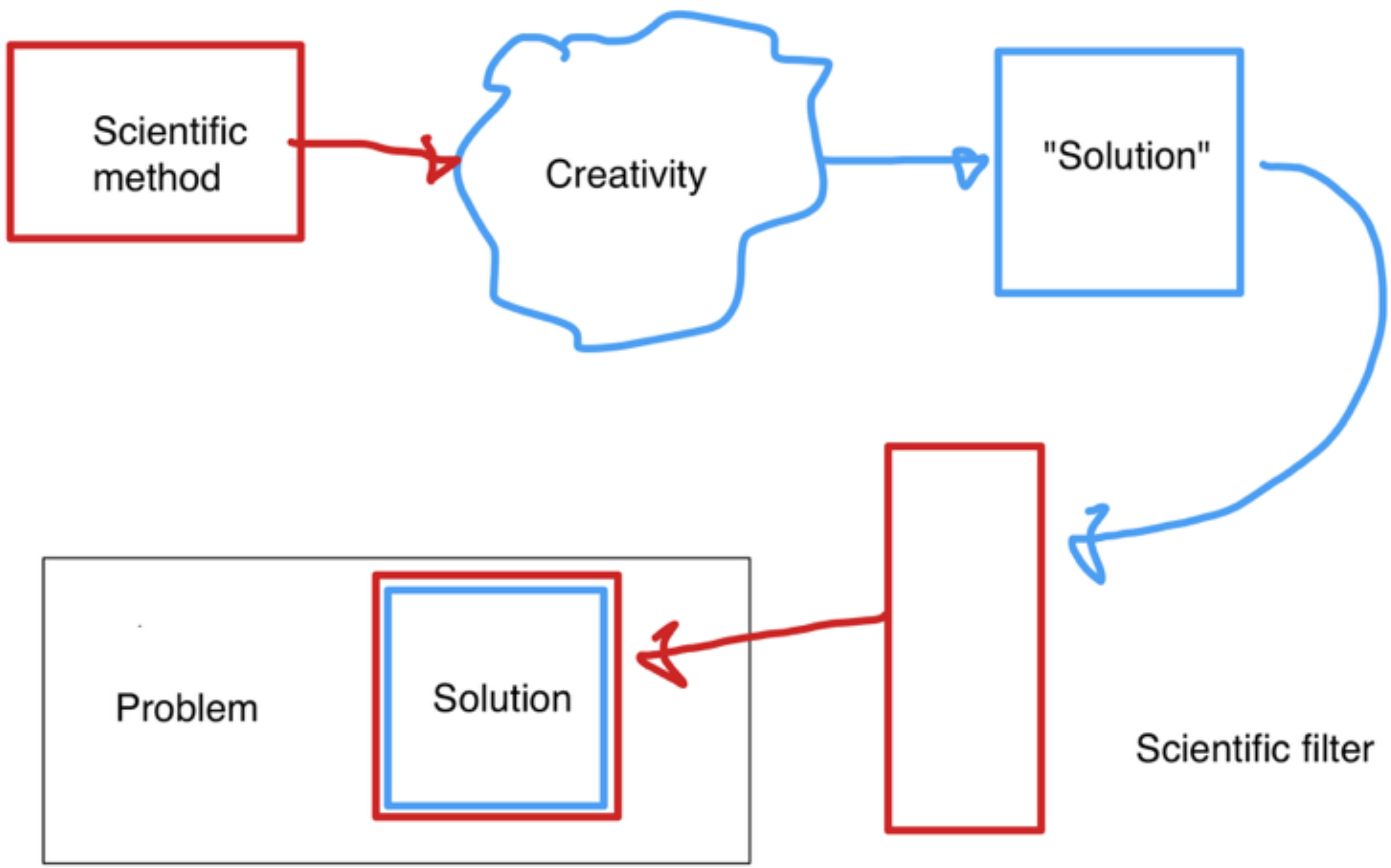
- The person working in this way may have to create and identify original problem solutions every step of the way. This will usually involve a variety of theories and methods, often ranging across more than one discipline since real-world problems are likely to be ‘messy’ and not soluble within the narrow confines of an academic discipline.

Science in an engineering project

The ordinary engineering process



The process with science "added"



What is the scientific filter?

1. We must put our solution in a broader scientific context. We must give references to other solutions and similar problems.
2. We must prove scientifically that our solution is *correct.*
3. We must put our solution in form of a report following scientific standards.

General questions

- What do you want to do? What is your project?
- Why do you want to do it? Is it important? Is it interesting?
- How do you plan to do it? Which methods will you use?
- When do you plan to do it? How long time will it take?

The subject

- It should be clearly stated.
- It should be significant. For instance, it should not be just a repetition of something already done.
- It should have clearly stated boundaries.
- It should be such that relevant data can be obtained.
- It should be such that significant conclusions can be drawn.

The form of the project

- It can be in form of a question, for instance, is functional programming better than imperative programming?
- It can be in the form of an hypothesis, for instance, functional programming is better than imperative programming.

The importance of being right

A famous mathematician once said that the most important thing is being right. You must have the talent for choosing hypotheses that are correct. You must have a sound intuition!

Scientific method in project work

We can characterize the project work by dividing it into four phases:

- Preparing Analysis
- Finding hypotheses
- Synthesis of partial results
- Validation of results

Analysis

The goal is to get an understanding of the problem/project. This understanding can involve the following steps:

- Describe the problem.
- Decide on a measure of success.
- Do studies on similar problems.
- Define goals.

Hypothesis

Here we have to be creative and try to find hypotheses and possible solutions to problems. This includes:

- State the hypothesis/solution clearly.
- Find consequences of the hyptothesis/ solution.
- Find criteria for judging if the hypothesis/ solution is true/works.

Synthesis

Here we test the hypothesis or implement and test the solution:

- If we have a solution to a problem we implement the solution.
- Do experiments for testing if the consequences of the hypothesis are true or if the solution works.
- Analyze the results.

Validation

Here we evaluate the hypothesis/solution and the results of the experiments:

- Try to measure how well the experiments confirm/falsify the hypothesis or how well the solution works.
- Try to decide if the hypothesis is true or if the solution works.
- Do documentation by writing a rapport or scientific paper.
- Submit your results for criticism from colleagues or independent referees.

Deductive systems

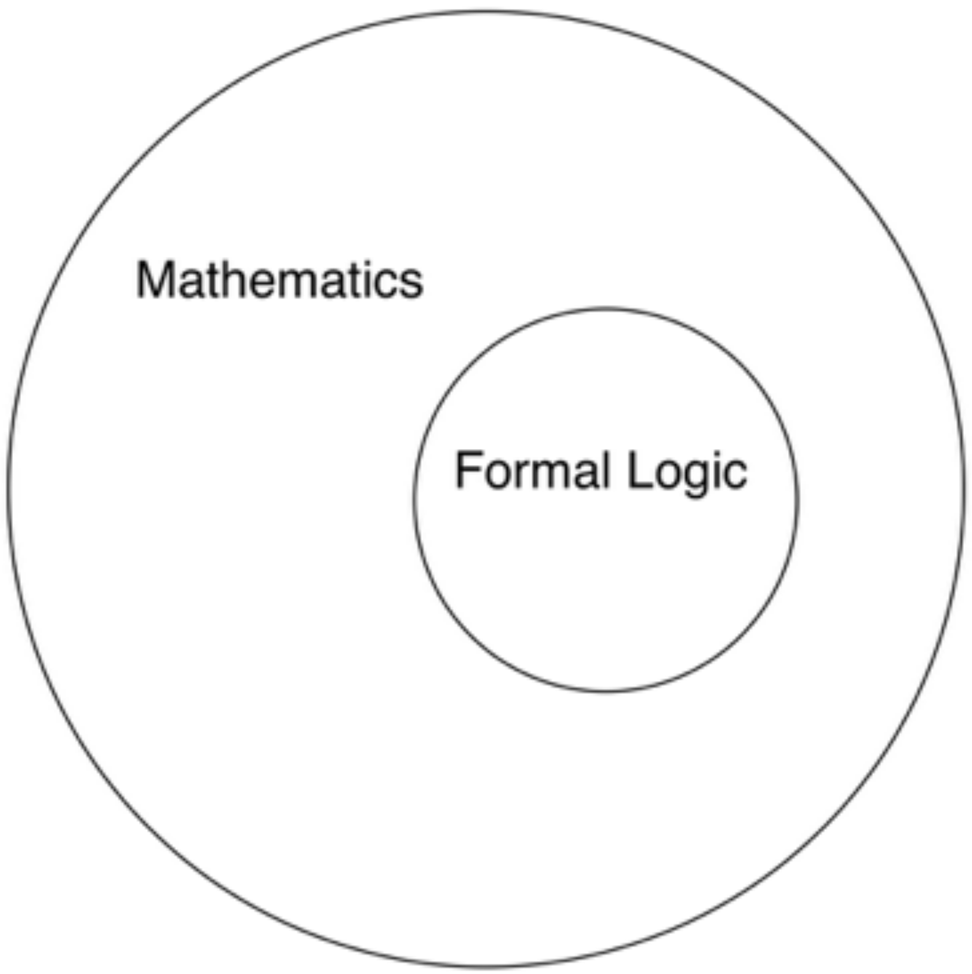
Users of formal systems

- Mathematicians - use them to prove mathematical theorems.
- Computer scientists - use them to design algorithms that solve problems.
- Philosophers - use them to define and analyze things.

Mathematics and Formal Logic

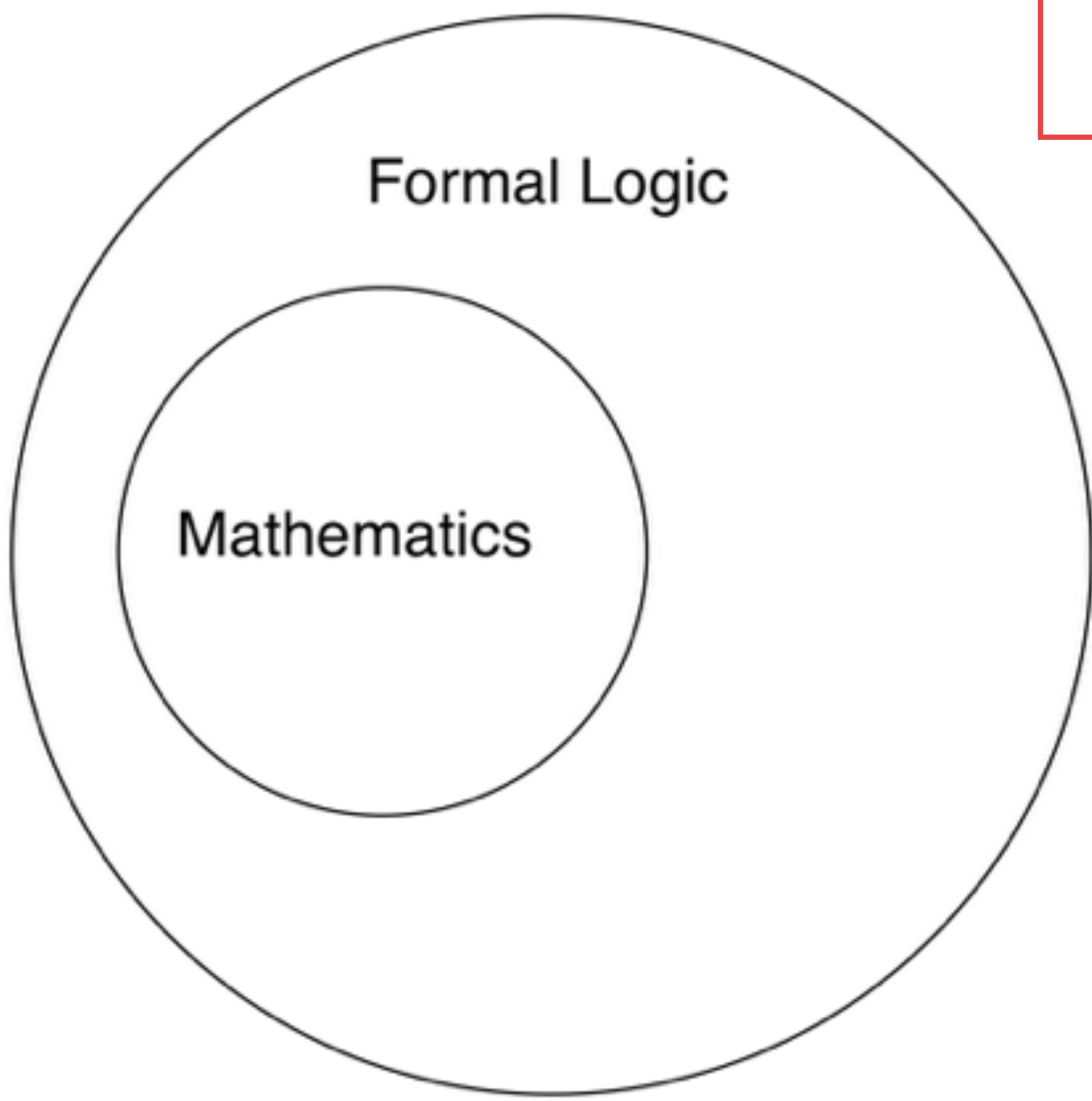
What is the connection between Mathematics and Formal Logic? Here are some suggestions:

Formal Logic is a part of Mathematics



This would probably be what mathematicians think

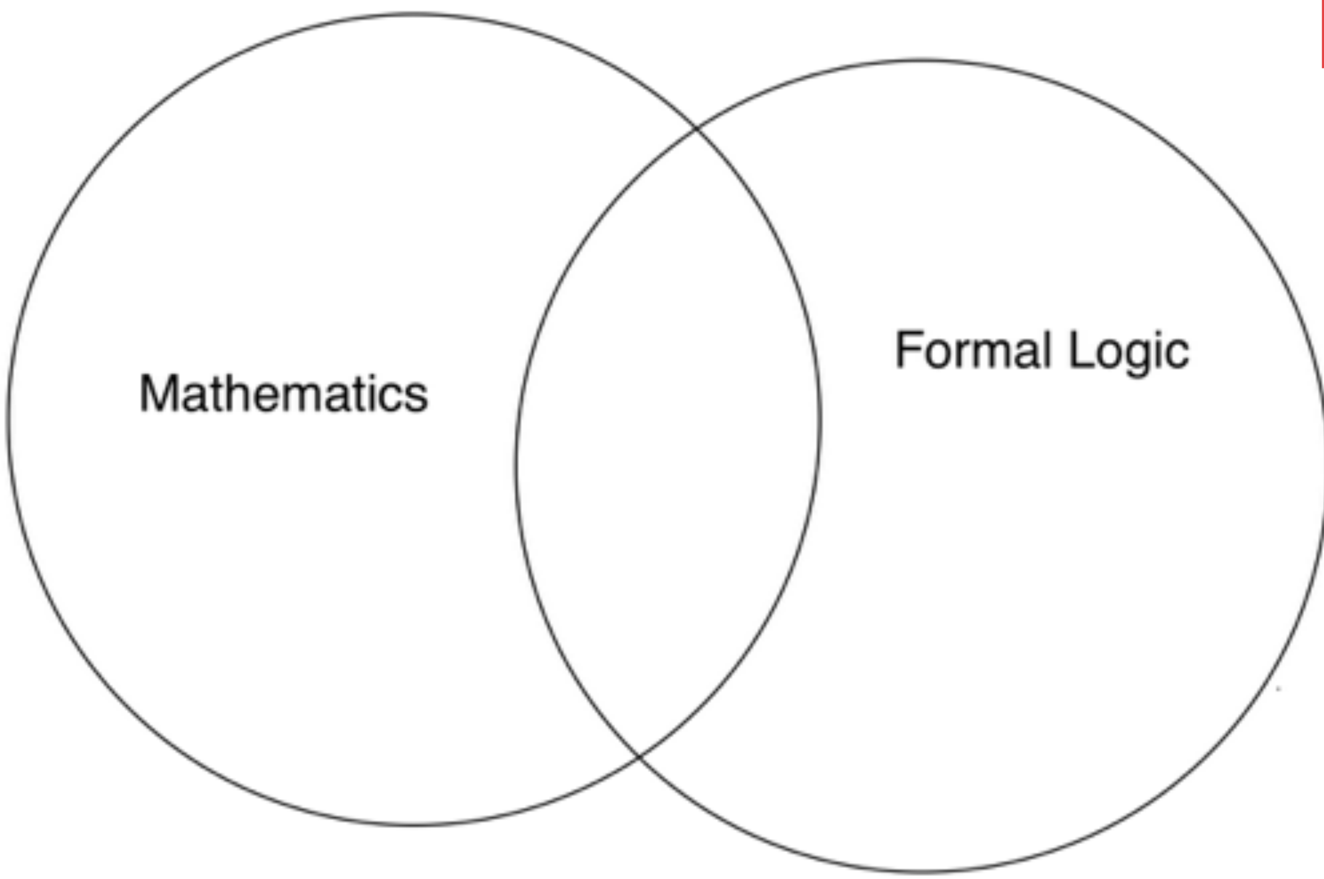
Mathematics is a part of Formal Logic



This is what the pioneers in Formal Logic thought

Neither is a part of the other

Nowadays, this seems to be natural view



Three components of a deductive system

- Vocabulary
- Deduction Rules
- Axioms

Vocabulary

We will look at some text from different disciplines all using formal syntax. It is normally rather easy to recognize the discipline.

Mathematics

Corollary 10.82 (Künneth Formula for Homology, II). *Let R be a right hereditary ring, let $(\mathbf{A}, \delta)$ be a complex of projective right R-modules, and let $\mathbf{C}$ be a complex of left R-modules.*

(i) *For all $n \geq 0$, there is an exact sequence*

$$\bigoplus_{p+q=n} H_p(\mathbf{A}) \otimes_R H_q(\mathbf{C}) \overset{\alpha}{\rightarrowtail} H_n(\mathbf{A} \otimes \mathbf{C}) \overset{\beta}{\twoheadrightarrow} \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(\mathbf{A}), H_q(\mathbf{C})),$$

where $\alpha_n \colon \sum_p \operatorname{cls}(b_p) \otimes c_{n-p} \mapsto \sum_p \operatorname{cls}(b_p \otimes c_{n-p})$, and both λ_n and μ_n are natural.

(ii) *For all $n \geq 0$, the exact sequence splits:*[8]

$$H_n(\mathbf{A} \otimes_R \mathbf{C}) \cong \Big[\bigoplus_{p+q=n} H_p(\mathbf{A}) \otimes_R H_q(\mathbf{C}) \Big] \oplus \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(\mathbf{A}), H_q(\mathbf{C})).$$

Theoretical physics

Schrödinger's Equation

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t)$$

i is the imaginary number, $\sqrt{-1}$.

$\hbar$ is Planck's constant divided by 2π: 1.05459×10^{-34} joule·second.

$\psi(\mathbf{r},t)$ is the wave function, defined over space and time.

m is the mass of the particle.

∇^2 is the Laplacian operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Formal Logic

Ga	*1, 2, DM*
(~Fa → Ga)	*2-3, CD*
<u>~Ga</u>	***Hypothesis***
Fa	*1, 5, DM*
(~Ga → Fa)	*5-6, CD*
((~Fa → Ga) ^ (~Ga → Fa))	*4, 7, Conj*
((Fa v Ga) → ((~Fa → Ga) ^ (~Ga → Fa)))	*1-8, CD*
<u>((~Fa → Ga) ^ (~Ga → Fa))</u>	***Hypothesis***
<u>~(Fa v Ga)</u>	***Hypothesis***
~Fa	*11, DM*
(~Fa → Ga)	*10, Simp*
Ga	*12, 13, MP*
~Ga	*11, DM*

Computer Science

$\overline{\quad \rightsquigarrow a}$ EtaR

$\overline{x.(x \bullet k) \rightsquigarrow k}$ EtaL

$\overline{\quad \rightsquigarrow [k/\alpha]s}$ BetaR

$\overline{a \bullet x.(s) \rightsquigarrow [a/x]s}$ BetaL

$\overline{\mathrm{not}[a] \rightsquigarrow a \bullet k}$ BetaNeg

$\overline{inl\ a \bullet [k, l] \rightsquigarrow a \bullet k}$ BetaCoPr

$\overline{, l] \rightsquigarrow a \bullet l}$ BetaCoProd2

$\overline{<a, b> \bullet \mathrm{fst}\ k \rightsquigarrow a \bullet k}$ BetaPr

$\overline{\bullet \mathrm{snd}\ k \rightsquigarrow b \bullet k}$ BetaProd2

Chemistry

$$2\mathrm{NaX} + \mathrm{Ca}^{2+} \leftrightarrow \mathrm{CaX}_2 + 2\mathrm{Na}^{+} \quad \log K_{\mathrm{NaCa}} = -0.20$$

$$K_{\mathrm{NaCa}} = \frac{\beta_{\mathrm{Ca}}[\mathrm{Na}^{+}]^2}{\beta_{\mathrm{Na}}^2 \gamma_0^2 [\mathrm{Ca}^{+}]} = 0.62$$

$$\mathrm{NaCl}_{(aq)} \leftrightarrow \mathrm{Na}^{+} + \mathrm{Cl}^{-} \quad \log K_{\mathrm{NaCl}} = 0.78$$

$$K_{\mathrm{NaCl}} = \frac{\gamma_0^2 [\mathrm{Na}^{+}][\mathrm{Cl}^{-}]}{\left[\mathrm{NaCl}_{(aq)}\right]} = 6.0$$

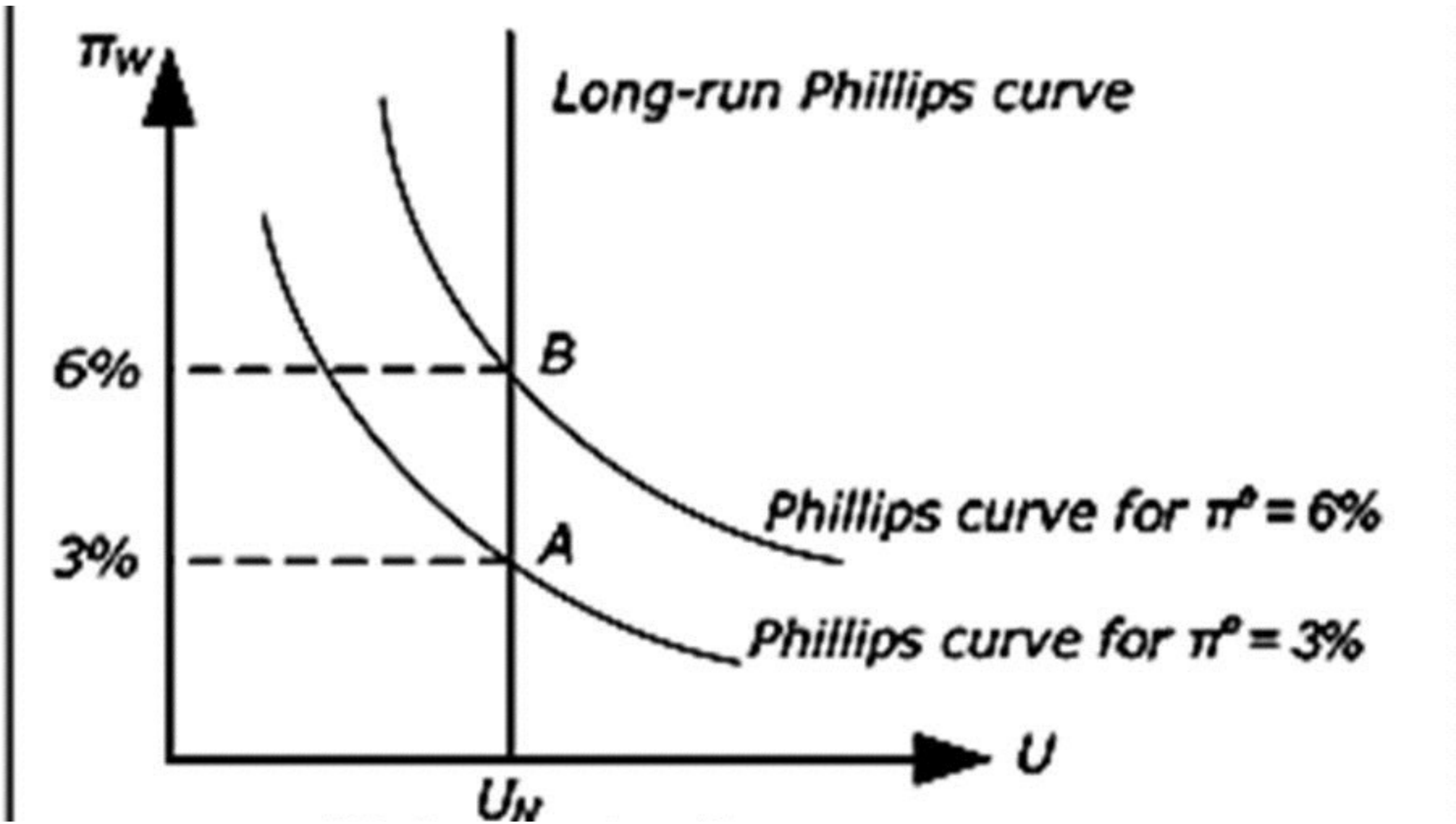
$$\mathrm{CaCl}^{+} \leftrightarrow \mathrm{Ca}^{2+} + \mathrm{Cl}^{-} \quad \log K_{\mathrm{CaCl}} = 0.79$$

$$K_{\mathrm{CaCl}} = \frac{\gamma_0^4 [\mathrm{Ca}^{2+}][\mathrm{Cl}^{-}]}{[\mathrm{CaCl}^{+}]} = 5.0$$

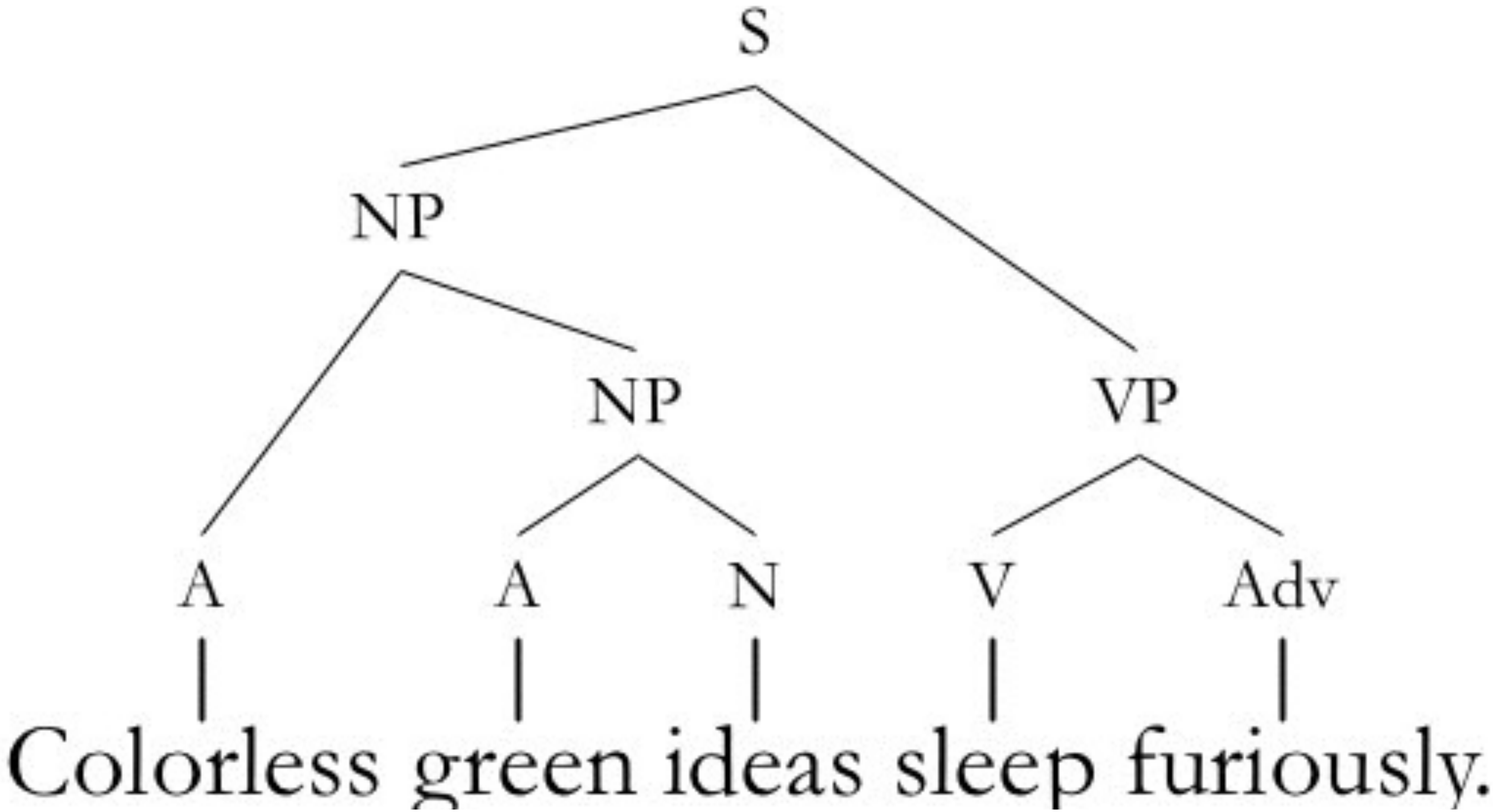
$$c_{\mathrm{Na}} = [\mathrm{Na}^{+}] + \left[\mathrm{NaCl}_{(aq)}\right] + c_{\mathrm{NaX}}$$

$$c_{\mathrm{Ca}} = [\mathrm{Ca}^{+}] + [\mathrm{CaCl}^{+}] + c_{\mathrm{CaX}_2}$$

Mathematical Economics



Linguistics



Vocabularies

- In a deductive system the vocabulary is roughly the syntax of the language we use in the system.
- Less formally, we can say that the vocabulary defines the type of expressions you can expect to find in the system.
- For instance, in text on evolutionary theory you would expect to find words like *natural selection* and so on.
- In formal logic the vocabulary is defined in a very precise way.

Deduction rules

- All deduction systems have some set of formal and informal rules which tells us what conclusions we can prove from other statements.
- In physics the rules are somewhat informal and established by praxis.
- In formal logic the deduction rules are where precisely defined.
- In mathematics it can happen that the deduction rules are implicitly understood. They can, however, be exactly stated (one would hope?)

Axioms

- The main idea is that the axioms are basic truths (intuitive truths maybe).
- Starting with axioms and using the deduction rules we create theorems.
- The axioms and theorems are the only truths in the system.
- In formal systems we divide the axioms into logical and non-logical axioms.
- In some systems with very strong deduction rules we have no logical axioms at all. Natural deduction is one example.

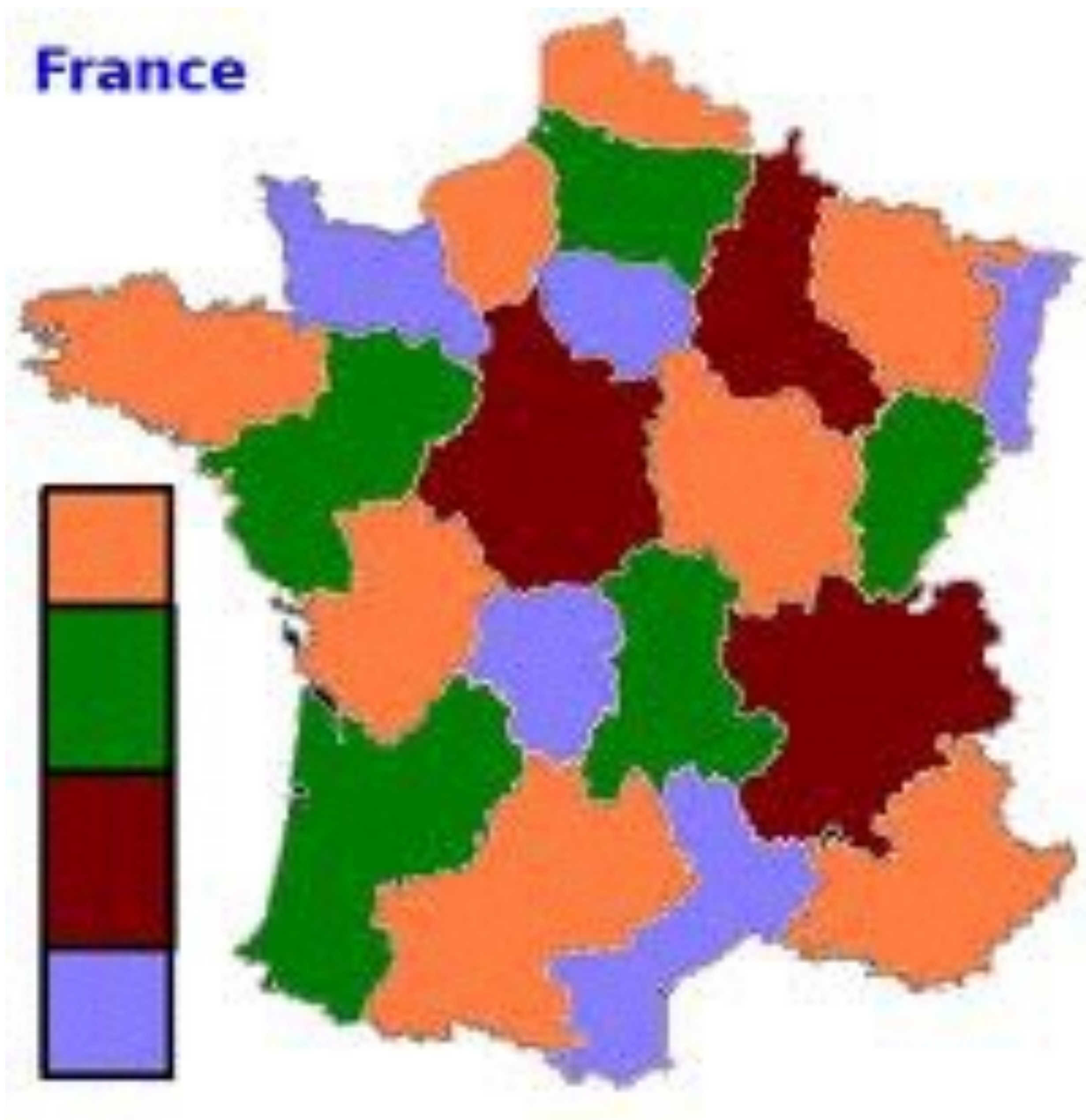
Do the axioms have to be true?

- The classic idea was that the axioms should be basic and fundamental truths.
- But later mathematicians realized that we could regard the axioms as assumptions and deduce consequences of these assumption.
- And important example of this is Non-Euclidian Geometries, developed in the 19th century.

Methodology?

- It seems to be very hard to give prescriptions for how research with deductive methods should be done.
- Its not that hard to learn techniques for checking if proofs are correct. The difficult thing is to find good theorems and theories.
- This is essentially a *creative* activity. And there are no recipes for creativity.
- Or are there? The best way of learning how to find proofs is to imitate existing proofs.
- Some other tricks will be described in a later lecture.

A case study: The Four-colour Theorem

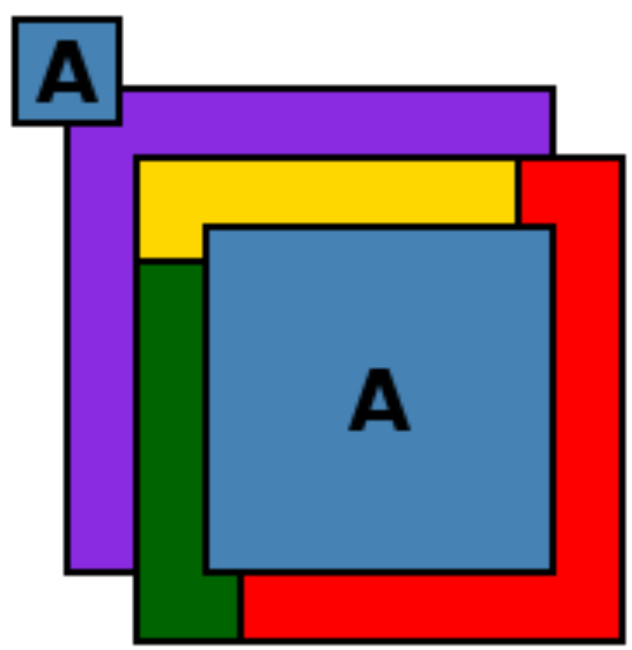


The theorem

- Every (planar) map can be coloured with four colours. A colouring is required to be such that no neigbouring countries have the same colour.
- This theorem was conjectured in 1852 and finally proved in 1976.

Is the theorem true?

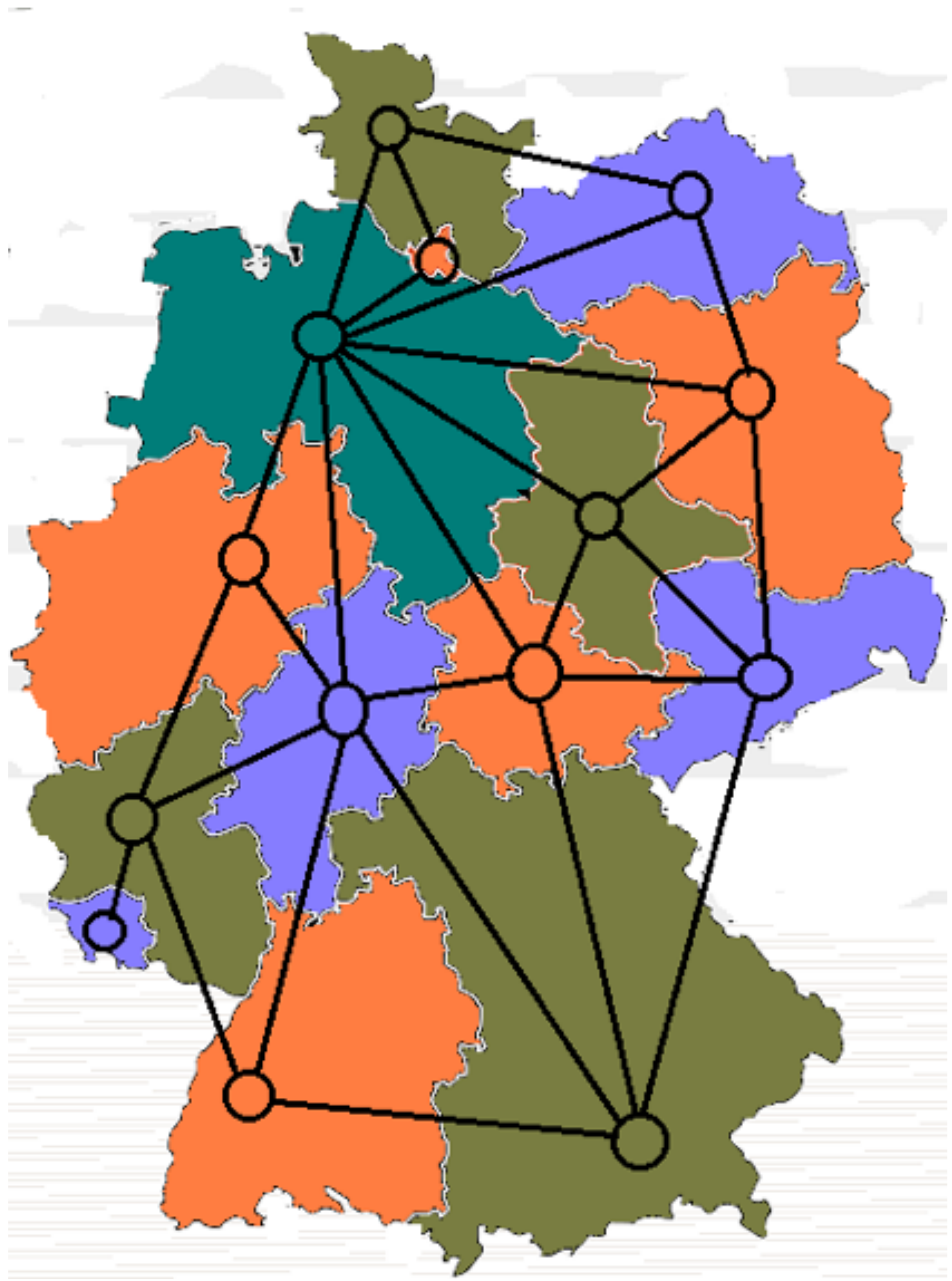
- If we look at a map we notice that two thing can complicate matters: Islands and lakes.
- More generally we see that non-connected countries will give us problems.
- It can be shown that if we allow non-connected countries we can find maps where the four-colour theorem is false.



A more exact formulation

- We require that the countries must be simply connected and have boundaries that are sufficiently simple.
- Furthermore, two countries meeting just in one point are not to be considered as neighbours.
- These complications make it natural to study the *dual* graph-form: Every planar graph can be coulored with four colours (in the normal node-colouring sense).

The dual graph

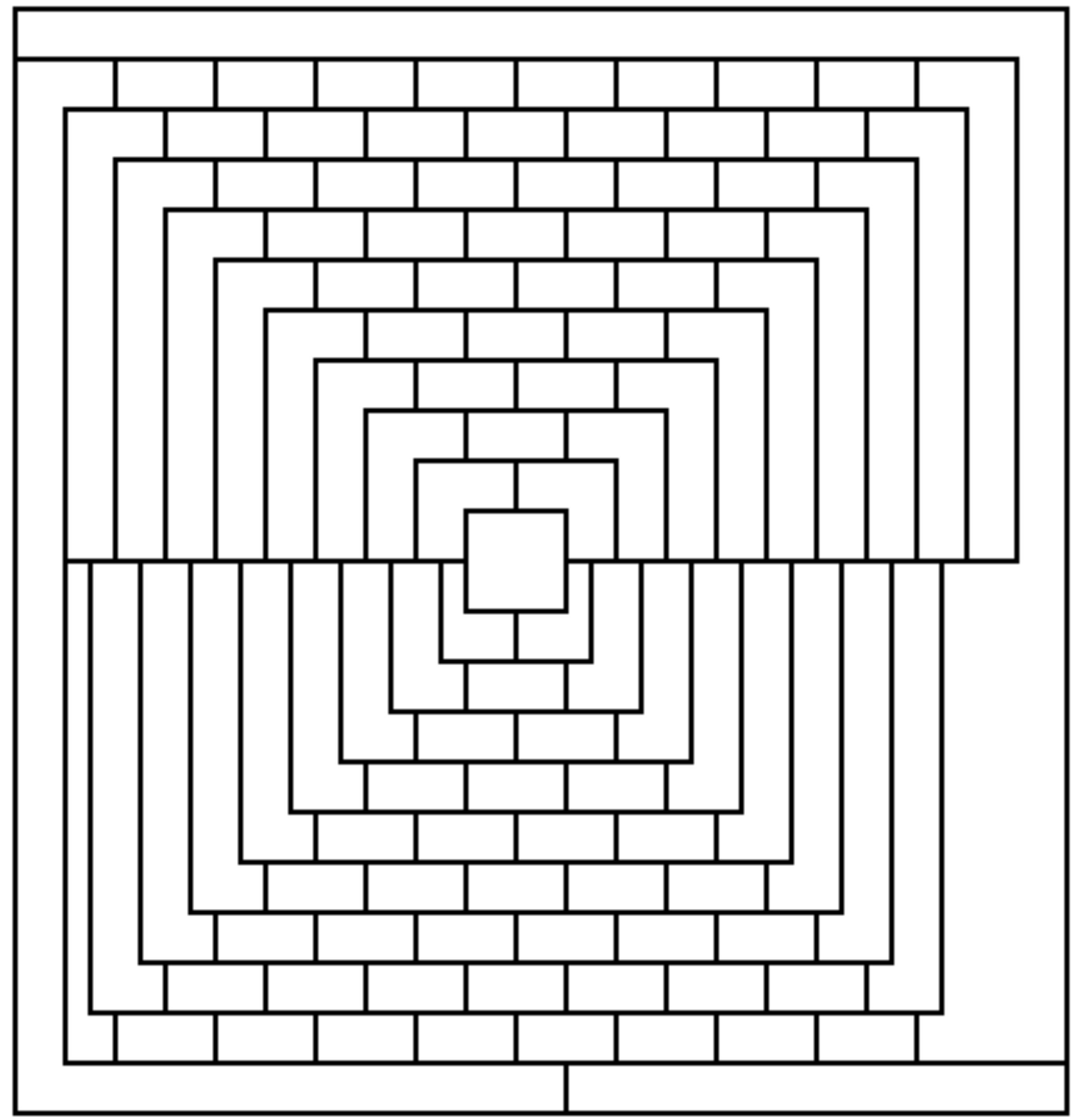


Method comments 1

- It is important to get an exact formulation of the problem as soon as possible.
- A problem can often be expressed in different forms. Even if the forms are equivalent, one of them can be easier to work with than the other.

True or false?

- When we face a conjecture we have to guess if it is true or not.
- If we think it is true we try to prove it.
- If we think it is false we try to find a counter-example.



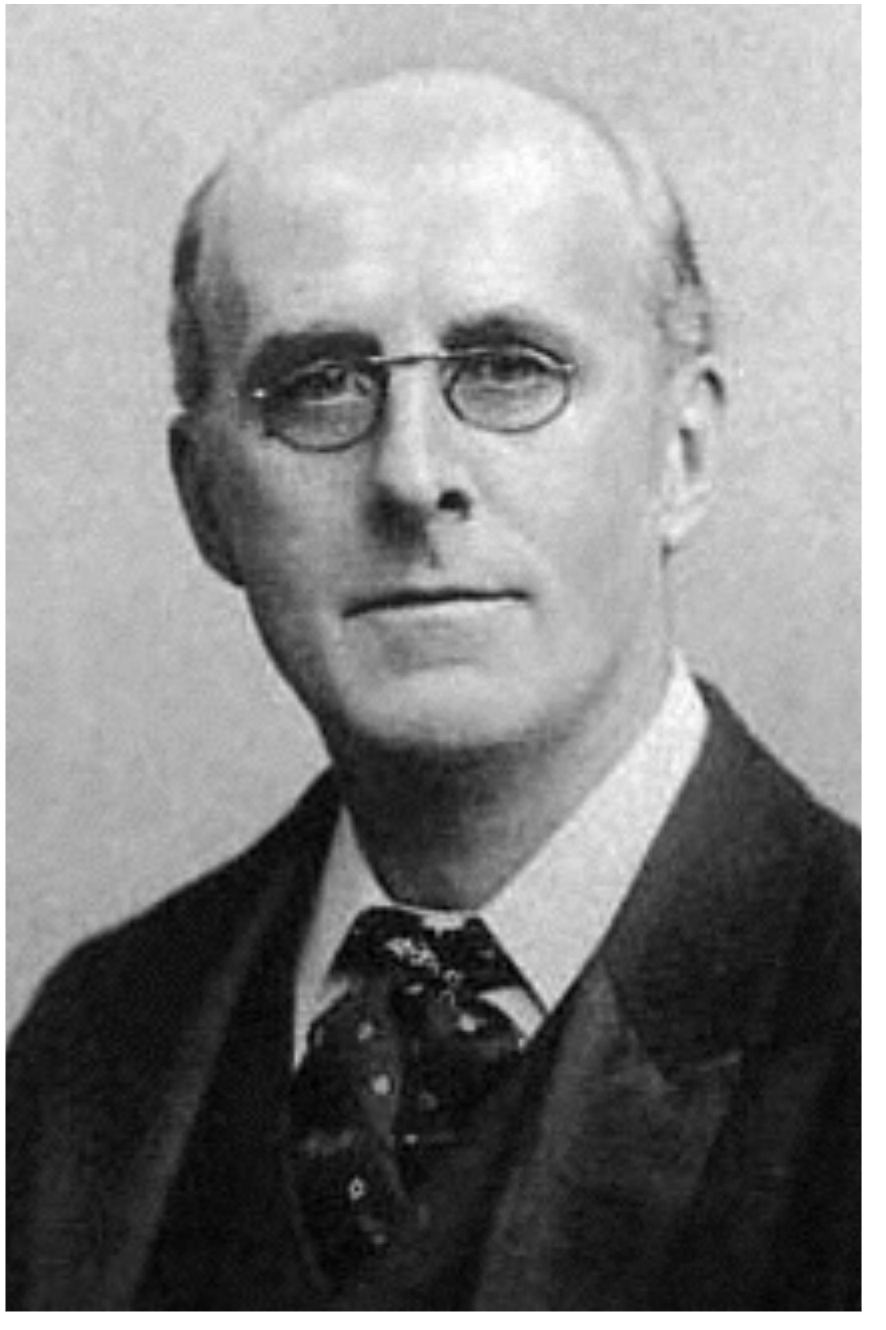
A counterexample?

A proof?

- Most people believed that the FC-Theorem was true. But how can we prove it?
- One attempt is to try to find an algorithm which actually colours any map with no more than four colours.
- But then we have to prove that the algorithm always manage to do this.
- We could try to find some more complicated existence-proof of a four-colouring.
- We could use mathematical induction.

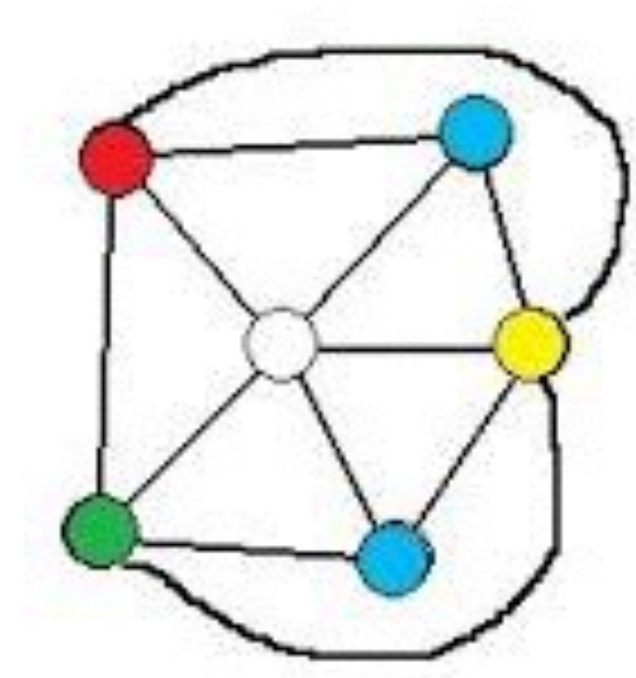
Kempe's "proof"

- In 1879 Sir Alfred Kempe managed to "prove" the FC-Theorem.
- He had a very good idea which used induction.
- He observed that all maps must contain at least one country surrounded with no more than five countries.



Details

- In the dual form we must have at least one node with degree no more than five.
- Remove the node and colour the rest of the graph with four colours.
- If, necessary, re-colour the graph so that no more than three colours are used around the start-node.
- Kempe "showed" that this can always be done.
- So then we can colour our graph with four colours!



Not so!

- In fact, the re-colouring which Kempe described does not work.
- This error was undiscovered for ten years!
- The error was then spotted by Heawood.

Method comments 2

- If a proof is erroneous, it means that there is a counterexample.
- Counterexamples come in two forms:
- Global counterexample - An example which shows that the statement in the theorem is false.
- Local counterexample - An example which shows that a step in the proof is incorrect.
- Kempe's proof fell due to a local counterexample (of course).

Algorithms

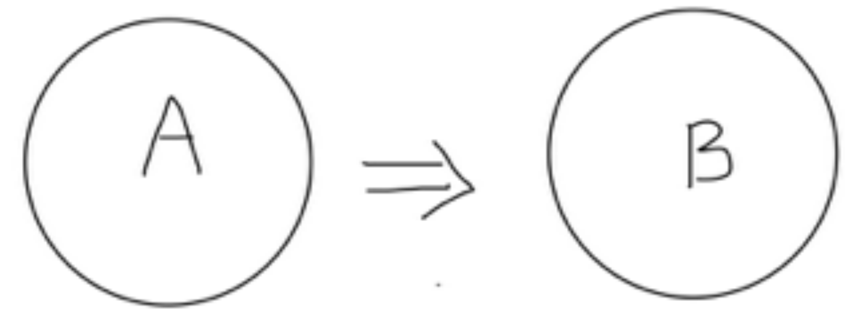
- We can apply the same reasoning to the correctness of algorithms.
- An algorithm takes an input and is supposed to deliver an output of a certain kind.
- An FC-algorithm take a plane graph as input and outputs a FC.
- We can speak of two kinds of counterexamples against the correctness of the algorithm:
- Global counterexample - An example which gives output on the wrong form.
- Local counterexample - An example which makes a certain step in the algorithm impossible to perform.

Method comments 3

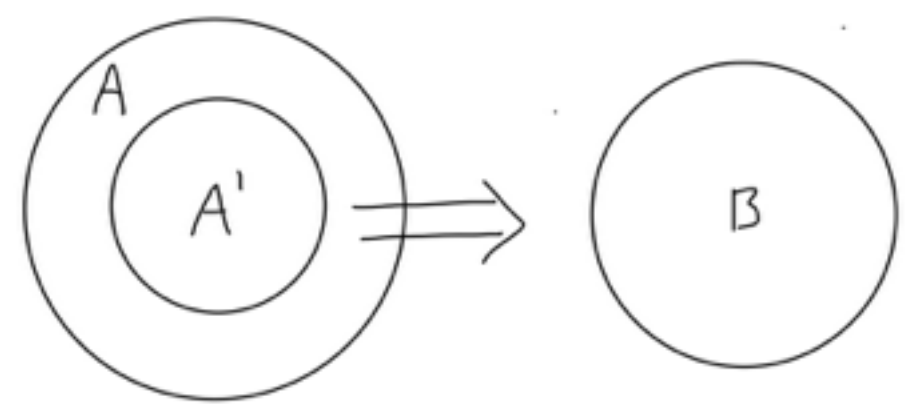
- Let us assume that we have a theorem of the form A => B. (For instance, A: A graph is plane B: The graph can be coloured with four colours.)
- We can *weaken* the theorem by replacing A or B with other statements. The weaker theorem can perhaps be proved.
- 1. Assume A' => A. Then A' => B is a *weakened* form of the theorem.
- 2. Assume B => B'. Then A => B' is a *weakened* form of the theorem.

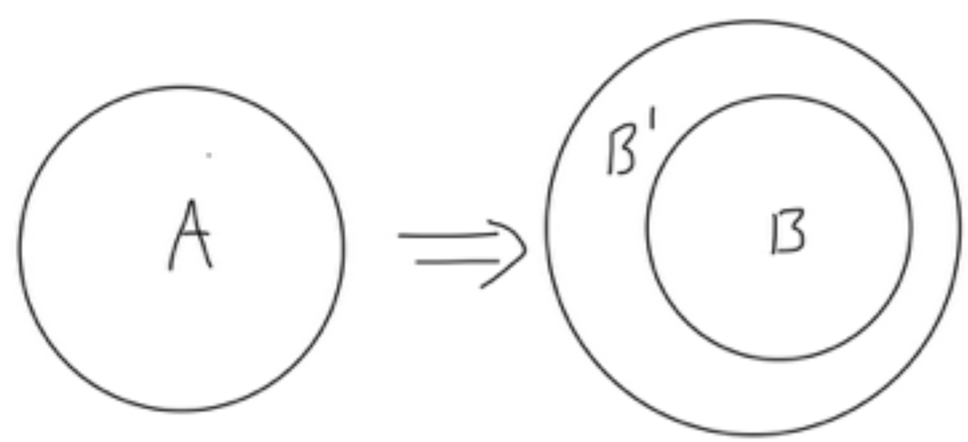
Weakening

The original theorem: A => B



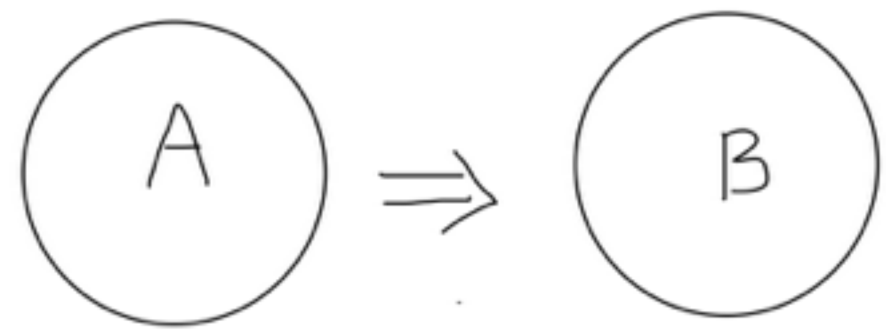
Weaker forms:



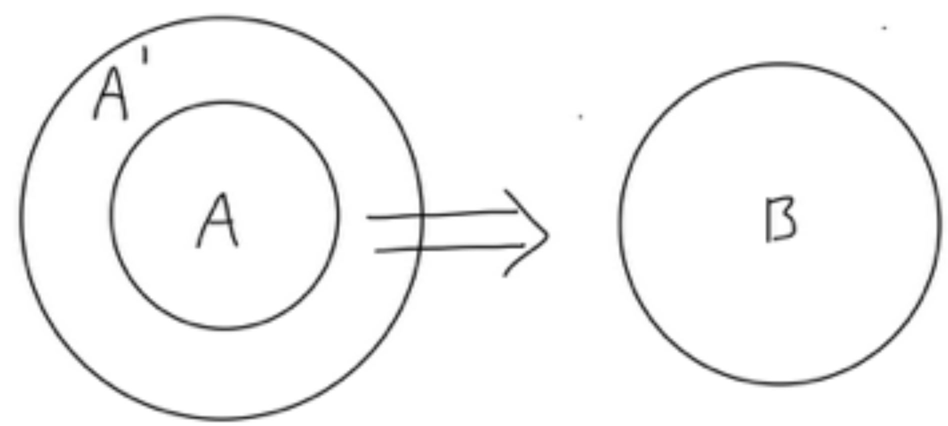


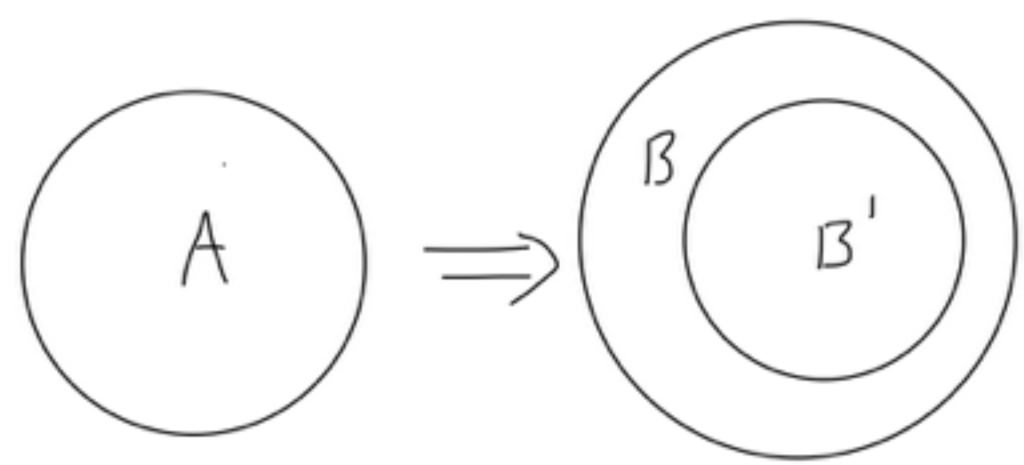
Strengthening

The original theorem: A => B



Stronger forms:



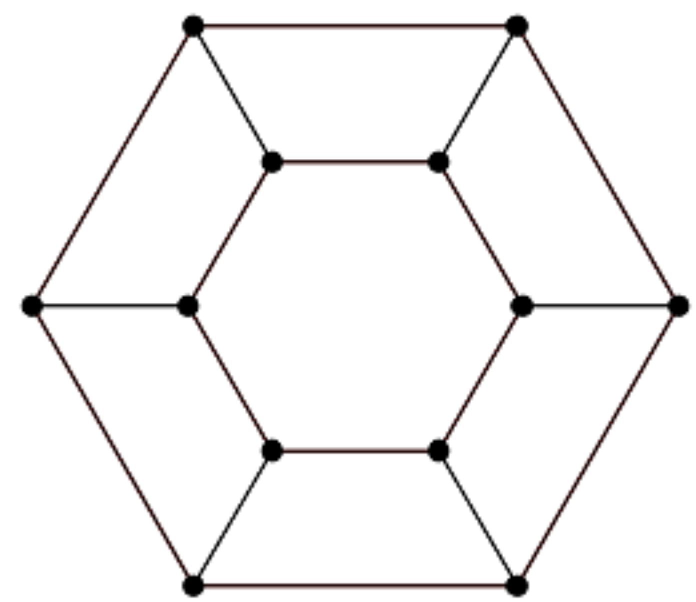


The Five-colour Theorem

- In 1890 Heawood used Kempe's technique and proved that every plane graph can be coloured with no more than five colours.
- It is obviously a weakening of the FC-Theorem.
- Heawood's proof shows that an erroneous proof (Kempe's) can still be useful.

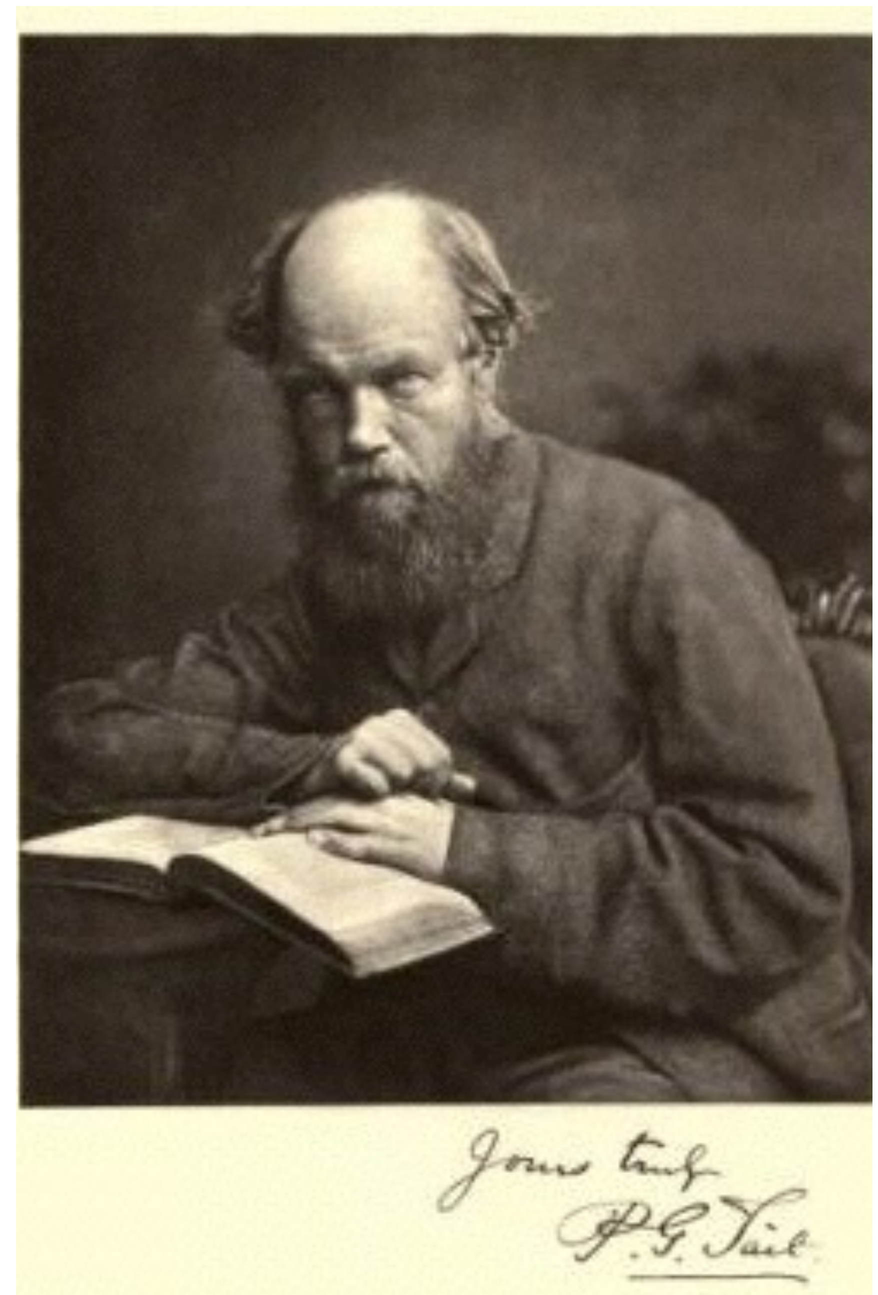
Another weakening

- Even before Kempe's proof it was known that it is enough to prove the FC-Theorem for *cubic* maps.
- Cubic maps - Maps where all nodes have degree three.



A reduction

- Tait managed to show that if we can show that every cubic map has a Hamiltonian Cycle, then the FC-Theorem must be true.
- But it turned out that there are (global) counterexamples to this statement, i.e. the existence of Hamiltonian Cycles.



A new idea: Edge-colourings

- Given a graph we can colour its edges. We say that a colouring is correct if any edges with a common node is coloured with different colours.
- Vizing's theorem: If N is the minimal number of colours needed to colour the graph G and D is the maximal node-degree in G, then N is either D or D+1.
- Tait showed that the FC-Theorem is true if and only if every plane bridgeless cubic graph can be edge-coloured with three colours.

Method comments 4

- We can speak about different problems. Informally we can say that Problem 2 is weaker than Problem 1 if a solution to Problem 1 would give us a solution to Problem 2.
- In the same way is Problem 1 stronger than Problem 2.
- And if a solution to any of the problems would give us a solution to the other one, we say that the problems are equivalent.

A comparison with Complexity Theory

- In complexity theory we have the notation ≤ where Problem 2 ≤ Problem 1 means that there is a polynomial time reduction from Problem 2 to Problem 1.
- In our more general discussion we do not have a *formal* definition of reductions in this sense.

What we have seen this far

- The problem of proving FCT for maps is equivalent to proving FCT to graphs.
- Heawood solved the weaker problem of proving that every plane graph can be 5-coloured.
- It was shown that FCT can be reduced to the (apparently weaker) problem of proving that every plane cubic map is three-colourable.
- Tait showed that FCT could be reduced to the problem of proving that every plane cubic graph has a Hamiltonian Cycle.
- Tait showed that FCT is equivalent to the problem of proving that every plane cubic graph can be edge-three-coloured.

Turning to harder problems

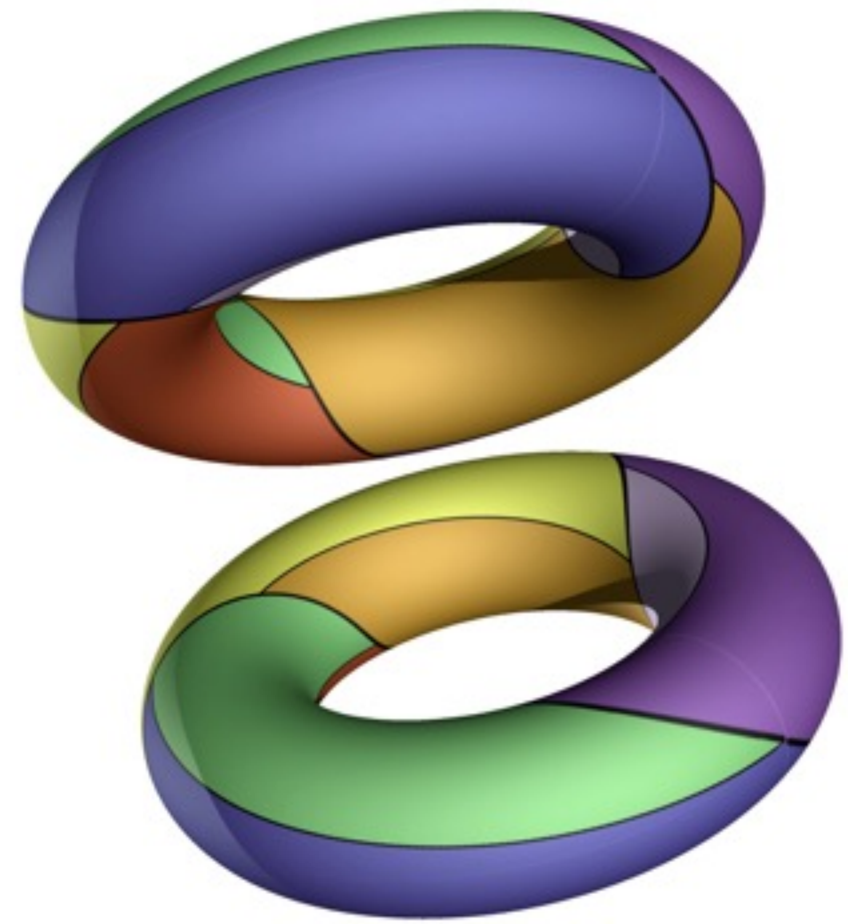
- It turned out that the FCT remained unproved despite all these promising approaches.
- What one could do then is to try to solve a harder problem.

Chromatic polynomials

- The mathematician Birkhoff tried to solve an apparently harder problem. He wanted to decide in how many ways an arbitrary graph G can be coloured with x colours.
- It turns out that the answer is a polynomial P(G,x), a so called *chromatic polynomial.*
- Birkhoff tried to show that for all planar graphs G we have $P(G,4) > 0$. But he didn't succeed.

Other types of maps

- Instead of plane maps we can consider maps on other bodies.
- For instance, on a torus it is quite easy to show that seven colours always suffice but not six colours.
- In fact, we can show variants of the FCT for all bodies except for spheres (which are equivalent to planes).



Method comments 5

- We have seen several promising attempts to prove the FCT. Eventually, none of them gave a proof.
- Nevertheless we see that trying to solve a problem can lead to other interesting problems and solutions to them.

The proof of the Four-colour Theorem

- The path towards the proof of the FCT starts with a return to Kempe's failed proof from 1879. The proof uses ideas that Kempe had.
- The proof uses induction over the size of the graph. Then we observe that a planar graph must have a set of *unavoidable* subgraphs.
- Then we prove that the subgraphs are *reducable*. This means that if the rest of the graph can be four-coloured, then this colouring can be extended to the subgraph with some minor changes to the original colouring.
- Kempe found the a simple unavoidable subgraph in form of a node with degree at most five. But he failed to prove that the subgraph is reducable (it is not).
- Appel and Haken had the idea that they should try to find more complicated unavoidable subgraphs.

A computer proof

- Appel and Haken managed to find a set of 1936 *together unavoidable* subgraph. (That means that in any planar graph at least one of the subgraphs must occur.)
- But in order to prove that the subgraphs were *reducable* they had to rely on a computer program to find the re-colouring strategies.
- The proof became much debated and criticized. It opened for a discussion of what a proof really is or should be.

Method comments 6

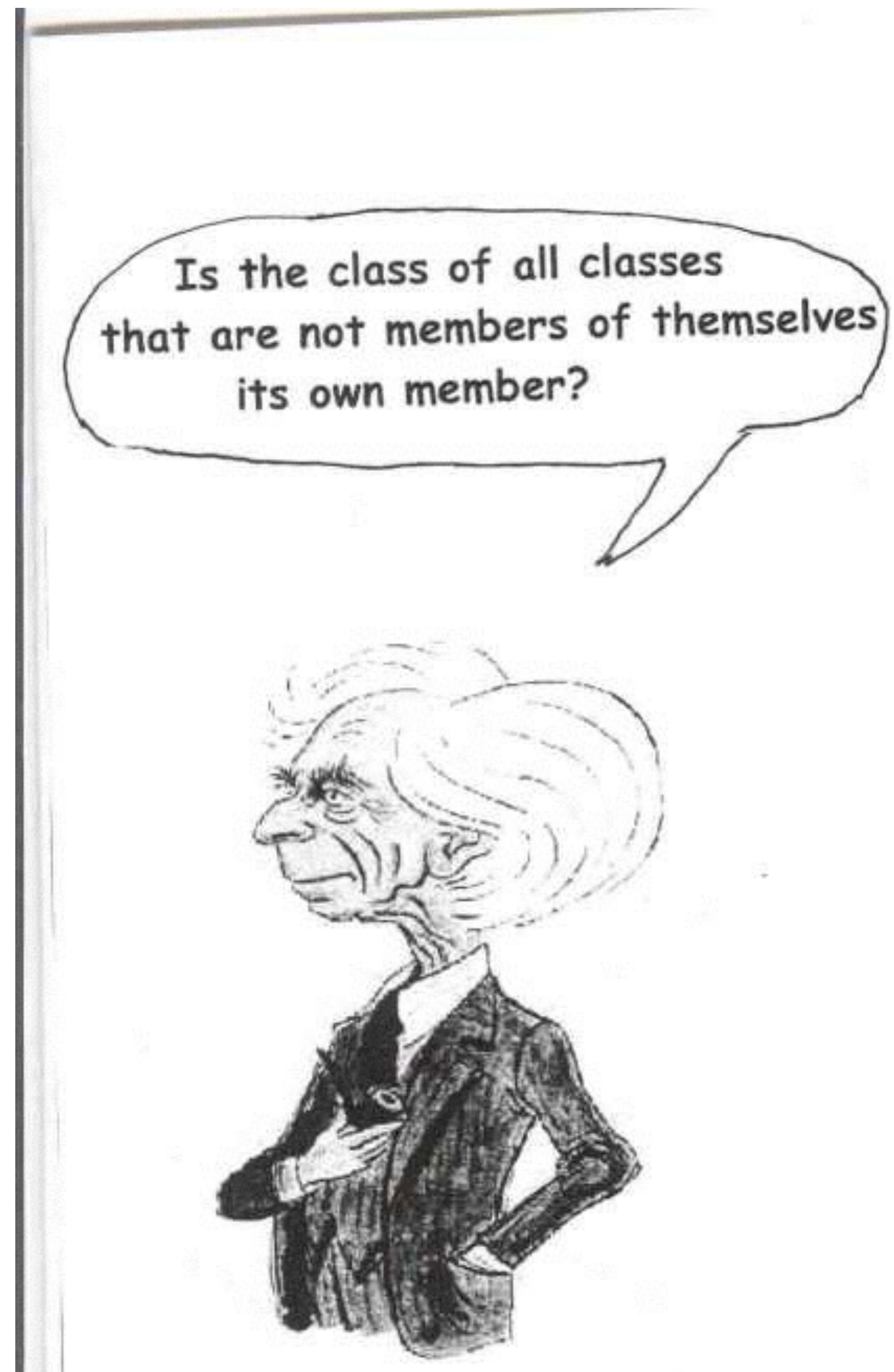
- So eventually the original idea by Kempe was triumphant.
- But in 1890 there was probably no easy way to see this.
- It was when all other strategies had failed that the return to the original idea seemed attractive.
- So sometimes a failed proof can be resurrected.

Paradoxes and impossibility theorems

- We will give a brief discussion of some problems and paradoxes related to deductive systems and mathematics.
- We will describe two great crisis in the history of logic and mathematics

Russell's paradox

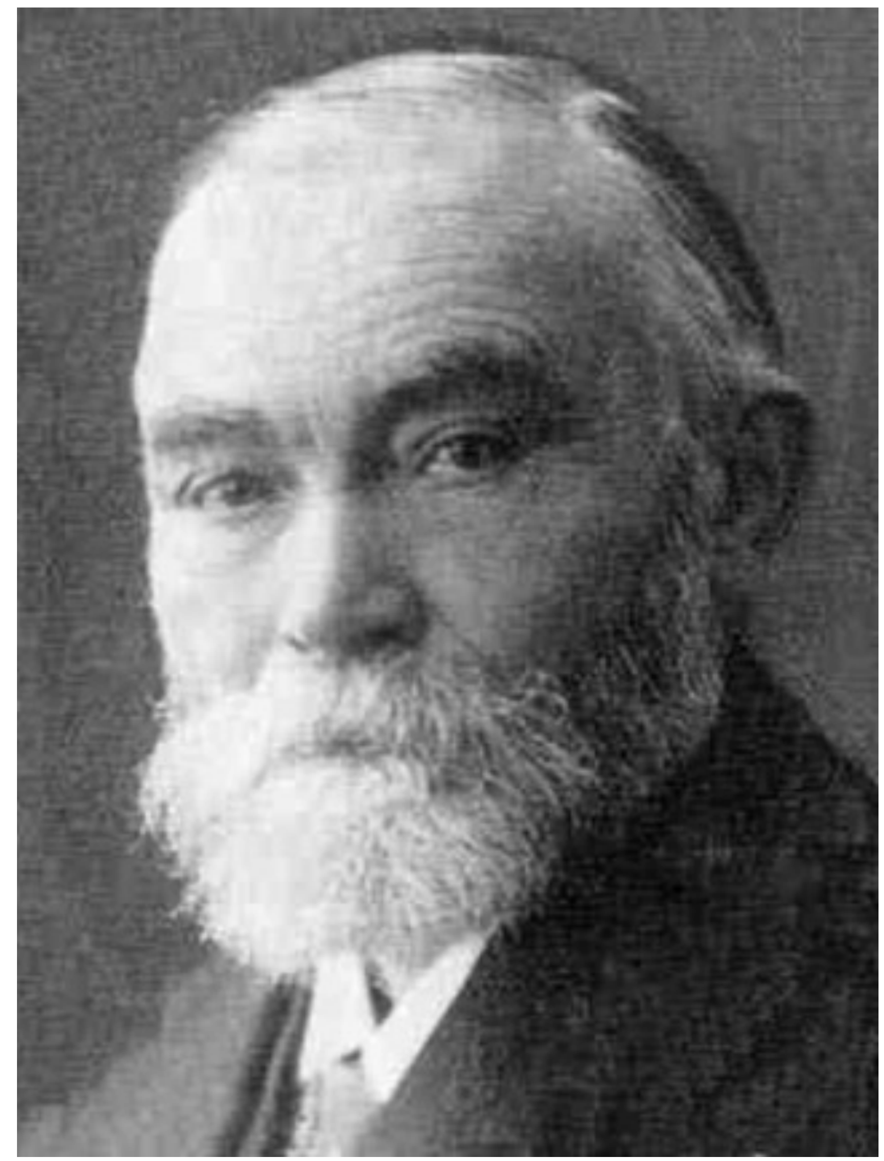
- The first crisis was in the early 20th century.
- We will start with some history.



Frege and mathematical logic

- Gottlob Frege created the modern mathematical logic at the end of the 19th century.
- He tried to construct all mathematics with logic.
- The starting point was a formalized version of set theory.
- Among other things Frege postulated that if P(x) is any predicate there always exists a set of all objects x such that P(x) is true:

$$\{x : P(x)\}$$



Bertrand Russell

- In the beginning of the 20th century Russell showed that Frege's axiom leads to contradictions.

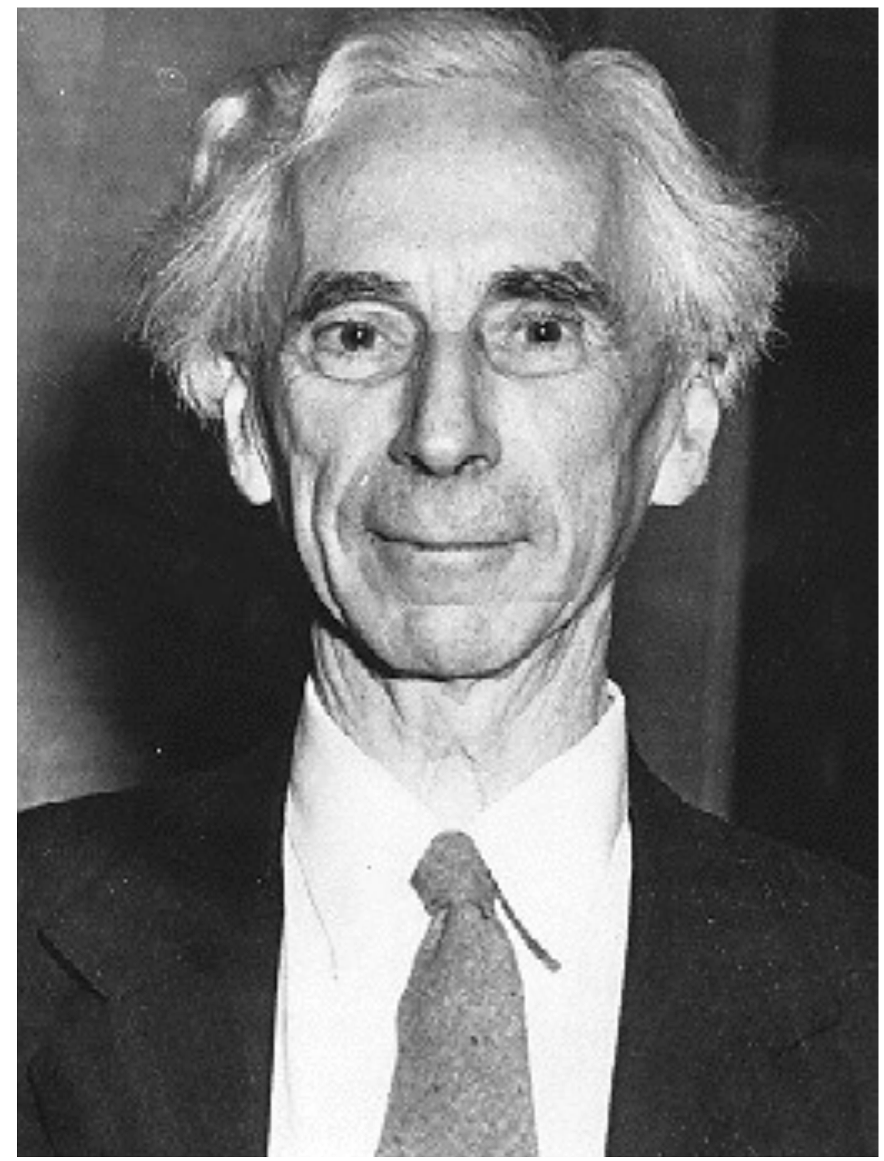
- If we define $P(x): x \notin x$

And $M = \{x : x \notin x\}$

What happens then? Is $M \in M$

or $M \notin M$

true?



Some related paradoxes

- The liar paradox - 'I am lying'. True or false?
- Grelling paradox - Among English adjectives there are some, such as 'short', 'polysyllable', 'English', which apply to themselves. Let us call such adjectives *autological;* all others are *heterological.* Thus 'long', 'monosyllable', 'green' are heterological. But what about 'heterological'? Is it heterological or not?
- Berry paradox - Consider the phrase "The smallest positive integer not definable in under eleven words". There must be such an integer (why?). But this integer is definable in ten words!

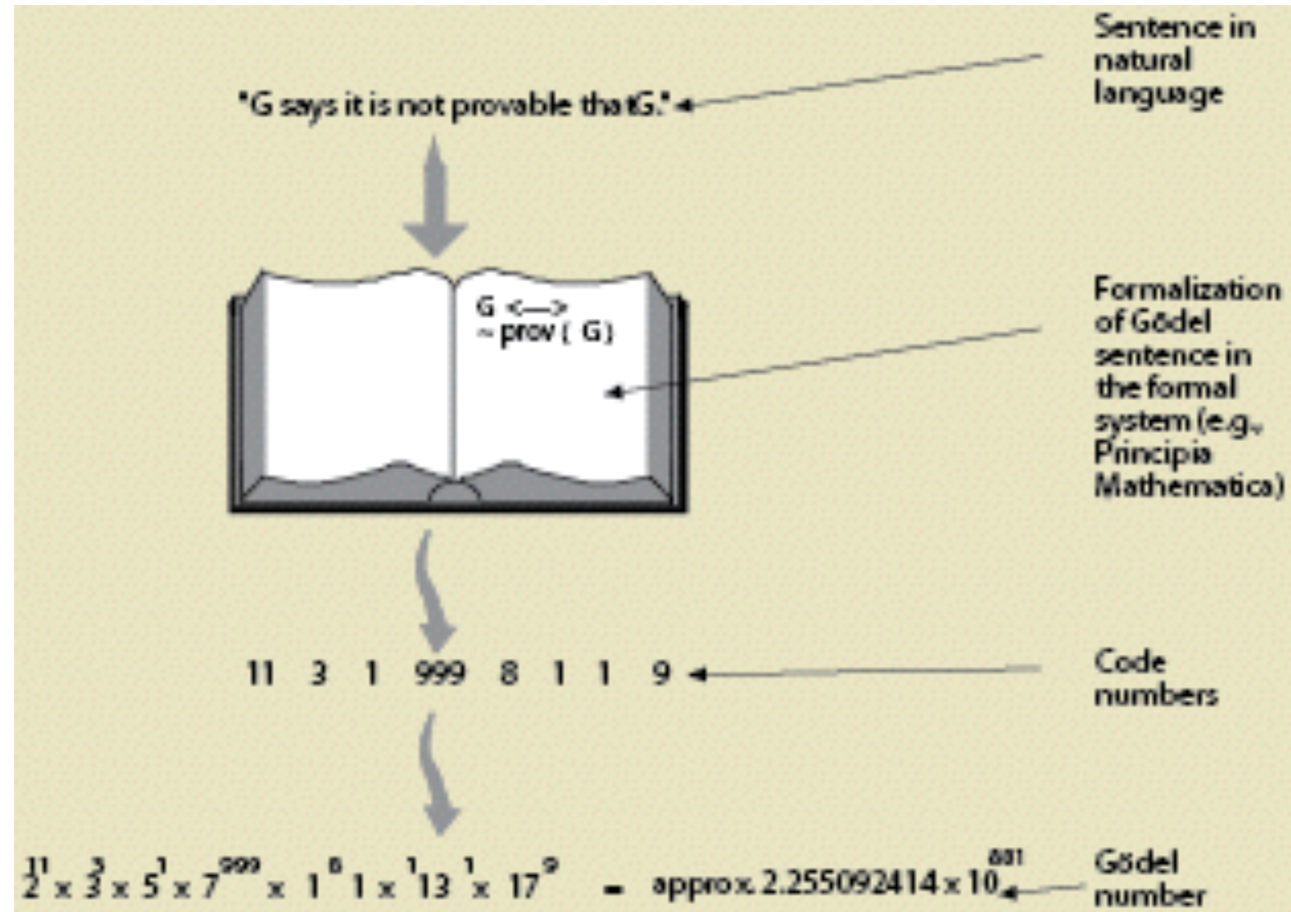
Russel's solution

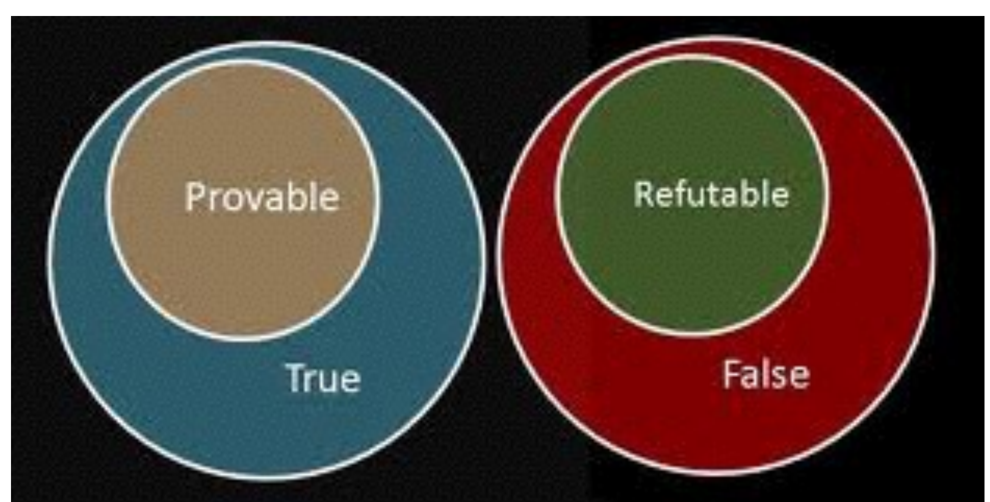
- Russell found that Frege's axiom must be restricted in some way.
- His idea was to block the possibility that a set could be a member of itself.
- In order to do that he developed the so called *type theory* of sets.
- Other solutions came soon. The paradox is not considered a problem any more.
- But a disturbing fact is that Frege was one of the greatest logicians ever and he felt that his axiom was (intuitively) obvious. If he could make such a mistake, can we ever be certain that we don't make similar logical mistakes?

The ghost of self-reference

- Frege's problem was that an unexpected self-reference occurred.
- An analysis of the other paradoxes seem to show that the also are the victims of self-reference.
- Suggestion: All these paradoxes are in some sense caused by self-reference.
- So if we just somehow can *block* all self-references there will be no paradoxes. Or ... ?

Gödel's Theorem





Gödel

- Kurt Gödel studied formal deductive systems of a special kind.
- He showed that all formulas in such a system can be given a so called *Gödel number.*
- He also showed that it is possible to construct a predicate that represents *provability.*
- Then he showed that there are sentences that cannot be proved in the system but still, in some more general sense, are true.

More details

- The Gödel Sentence: $G \leftrightarrow \neg Pr[S](\mathbf{G})$
- Gödel's theorem can be stated in at least two different forms.
- One form is that a sufficiently strong and (efficiently) decidable formal system must contain 'true' sentences which cannot be proved inside the system.
- Another form is that such a system must contain sentences which cannot be proved or disproved inside the system.
- To make things more complicated, there is a *Gödel's second incompleteness theorem* which says that the system cannot be proved to be consistent with methods inside the system.

Implications

- One thing Gödel's proof shows is that self-reference cannot actually be blocked. It is in a certain sense unavoidable.
- It also shows that the powers of formal systems are limited.
- We could of course accept these facts.
- Or we could just give up the idea of using formal systems.
- There are however some related theorems which are even more disturbing.

Tarski

- Alfred Tarski showed that the definition of truth is much more complicated than expected.
- The Tarski type of truth definition is like this: 'Snow is white' if and only if snow is white.
- This type of definition requires a *meta-level.* Truth comes in *layers,* so to say.
- And there is no way to define truth in any effectively decidable way.

Turing

- As we all know, Alan Turing defined the Turing Machine.
- He proved that there are natural problems which cannot be solved in an 'mechanical' way.
- An example is the halting problem.
- Another is the problem of finding proofs in first order logic.