Matematik, KTH **SF2740, Graph Theory**
Svante Linusson **HT 2015**

Second set of hand in problems

Be sure to write solutions with clear arguments that are easy to follow. You should try to have a level of details so your solution would be understandable to other students. Staple your solution together in the top left corner and write down your solutions in order. Write your name in the top right corner.

Code of conduct(Hederskodex): It is assumed that:
-you shall solve the problems on your own (or in cooperation with one or two fellow students) and write down your own solution. If you cooperate with someone you must mention that persons name for each problem.
- you must not use other resources
- if you in spite of this are using something you have gotten from somewhere else for some reason (a friend, a book or the internet etc.) you must give a reference to the source.

Your solutions to these problems are due November 20, before class starts.

(1) (10p) To the disjoint union of the graph $H := \overline{K^{2m+1}}$ and k copies of K^{2m+1} add edges joining H bijectively to each of the K^{2m+1}. Show that the resulting graph G contains at most $\kappa_G(H)/2$ independent H-paths. (This is exercise 3.22 in Diestel.)

(2) A graph is said to be **imbedded** on a (closed) surface if it is drawn on the surface with edges not intersecting (except at vertices) just like we did for plane graphs. For a surface S let $\chi(S)$ be the largest chromatic number of any graph imbedded on S. As for plane graphs we let F be the set of regions (faces))that the graph subdivides the surface into. An important invariant for a surface is the **Euler characteristic** c. E.g. the sphere has $c = 2$, the projective plane has $c = 1$, the torus has $c = 0$ and other surfaces have still smaller c. Compare to the genus $g \geq 0$ of an orientable surface with the relation $c = 2 - 2g$.

You don't need to know the definition of $c(S)$ but you will need the following lemma.

Lemma 1. *For a graph $G = (V, E)$ imbedded on S we have $|V| - |E| + |F| \geq c(S)$.*

The goal of this exercise is to prove the following theorem.

Theorem 2. *For any closed surface S with $c \leq 1$, $\chi(S) \leq \lfloor \frac{7+\sqrt{49-24c(S)}}{2} \rfloor$.*

We divide the proof into steps.

a) (5p) Prove

Lemma 3. *If G is imbedded on S, then $d(G) \leq 6 - \frac{6c(S)}{|V|}$.*

b) (5p) A graph G is called **special** if removing any edge lowers the chromatic number. Prove

Lemma 4. *Assume that G is imbedded on S, that G is special and connected, and that $\chi(G) = \chi(S)$. Then for all $v \in V(G)$ we have $d_G(v) \geq \chi(S) - 1$.*

c) (7p) Use the lemmas to prove Theorem 2 when $c \leq 0$. (Hint: prove non-negativity of a certain polynomial.)

d) (3p) Use the lemmas to prove Theorem 2 for the projective plane, i.e. when $c = 1$.

e) (1000000p) Bonus problem. Prove Theorem 2 for the sphere, i.e. when $c = 2$.

(3) (10p) Let G be a graph that has no induced subgraph isomorphic to $K_{1,3}$.
a) Let c be a vertex coloring of G. Prove that the induced subgraph on the union of two color classes is a disjoint union of cycles of even length and paths.
b) Let c be a vertex coloring of G using k colors. Prove that there is a coloring c' of G with k colors such that any two color classes of c' differ in size by at most one.
(hint for b): use a))

(4) (10p) Describe all graphs G with two non-adjacent vertices u, v such that:
(i) there are at most k independent $u - v$ paths in G
(ii) if any edge e is added to G there will be at least $k+1$ independent $u-v$ paths in $G \cup e$.
(Don't forget to also show why there are no others than the ones you describe.)

Lycka till!

Svante