MEETING 9 - INDUCTION AND RECURSION

We do some initial *Peer Instruction* ...

Predicates

Before we get into mathematical induction we will repeat the concept of a predicate. A predicate is a mathematical statement whose truth depends on a value of a variable. For example we could write:

$$A(n) \Leftrightarrow 8 | n^3 - n^2$$

and put in various values in n's place. For example, $n = 4$ gives the statement $A(4) \Leftrightarrow 8|4^3 - 4^2 \Leftrightarrow 8|64 - 16 = 48$ and since 8 does divide 48, the statement we get when putting 4 at the place of n becomes true. However, if we put $n = 5$ we get the statement $A(5) \Leftrightarrow 8|5^3 - 5^2 \Leftrightarrow 8|125 - 25 = 100$, and since 8 does not divide 100, the statement becomes false.

Mathematical Induction

The powerful idea behind the method of proof by mathematical induction can be illustratred by falling dominoes. If we line up a row of dominoes close to each other and tip the first one, they all fall.

If we wish to prove that some predicate is true for all integers, say $A(n)$ (as above) we can sometimes use this principle. If we can prove that $A(1)$ is true and also the the implication $A(p) \Rightarrow A(p+1)$ is true for all values of $p \geq 1$, then we have

$$A(1) \text{ true} \Rightarrow A(2) \text{ true} \Rightarrow A(3) \text{ true} \ldots$$

and so on forever, in short for all $n \geq 1$ we have $A(n)$ true. We will start by looking at a very simple example. Consider the following statement:

$$\text{the sum of the } n \text{ first odd natural numbers is } n^2.$$

Examples of this is $1 + 3 = 4 = 2^2$, that is the sum of the first two odd natural numbers is 4 which is 2^2, that is the square of 2. Another example is if we take the first 3 odd numbers and add them together: $1+3+5 = 9$ we get 9 which is the square of 3. And so on. We can formulate this with a predicate:

$$A(n) \Leftrightarrow \sum_{k=1}^{n} (2k - 1) = n^2$$

and we wish to prove that for all integers $n \geq 1$, $A(n)$ is true.

Theorem: $\forall n \in \mathbb{N} : \sum_{k=1}^{n} (2k - 1) = n^2$.

Proof: As we said above, for each n, we introduced the predicate $A(n)$ for the statement $\sum_{k=1}^{n}(2k-1) = n^2$. Then we wish to prove that $\forall n \in \mathbb{N} : A(n)$. A proof that relies on mathematical induction first involves a check that what we wish to prove is true for a starting value. Here the starting value is $n = 1$. Hence we wish to prove that $A(1)$ is true. Is it? Yes, since $A(1)$ can be formulated

$$2 \cdot 1 - 1 = 1^2$$

which is true. The left hand side is the sum of the first 1 odd numbers, which is just 1 itself, the right hand side is the square of 1 which is also 1, hence the left hand side equals the right hand side and so $A(1)$ is true.

The next step is to show that for all natural numbers p, the implication $A(p) \Rightarrow A(p + 1)$ is true.

To prove that an implication is true we need to suppose that the preqrequisite is true, this called the *Induction Hypothesis*. So we assume that for an particular natural number p we have $A(p) \Leftrightarrow \sum_{k=1}^{p}(2k - 1) = p^2 \Leftrightarrow LHS_p = RHS_p$. We have introduced a further notation here, we denote, by LHS_p the left hand side of the induction hyposthesis and by RHS_p the right hand side of the induction hypothesis. We now wish to prove that $A(p + 1)$ is true with this as a basis. That is we wish to prove that $LHS_{p+1} = RHS_{p+1}$. So we study LHS_{p+1} :

$$LHS_{p+1} = \sum_{k=1}^{p+1} (2k - 1) = \sum_{k=1}^{p} (2k - 1) + (2(p + 1) - 1) = LHS_p + (2p + 1).$$

Here is where we use the fact that the induction hypothesis holds, that is we use the equality $LHS_p = \sum_{k=1}^{p}(2k-1) = p^2 = RHS_p$ this means that this expression is in fact $p^2 + 2p + 1 = (p+1)^2$, but $(p+1)^2$ is, in it's turn equal to RHS_{p+1}! So we have

$$LHS_{p+1} = LHS_p + (2p+1) = p^2 + 2p + 1 = (p+1)^2 = RHS_{p+1}.$$

which means that $A(p+1)$ is true and since this followed from $A(p)$ we have established the truth of the implication $A(p) \Rightarrow A(p+1)$. Since $A(1)$ was true we conclude that $A(2)$ is true, and from this we get $A(3)$ and so on: that $A(n)$ must be true for all natural numbers n. That is $\forall n \in \mathbb{N} : A(n)$ which is what we wanted to prove.

The general outline of a proof that relies on mathematical induction to prove a statement of the form $\forall n \in \mathbb{N} : A(n)$ is the following:

1. Check that the statement holds for a starting value, above this starting value was 1. We checked that $A(1)$ was true and it was.
2. Prove the implication $A(p) \Rightarrow A(p+1)$. To prove this implication we first assume that $A(p)$ is true for a certain p. Then we use that information, that power given to us, to show that also $A(p+1)$ is true.
3. Lastly we appeal to the principle of mathematical induction where we write down the argument

$$A(1) \text{ is true } \Rightarrow A(2) \text{ is true } \Rightarrow A(3) \text{ is true } \Rightarrow \ldots \Rightarrow A(n) \text{ is true for all } n \in \mathbb{N}.$$

A proof resting on mathematical induction must contain all these three steps. They are each vital. We will study a number of examples of proofs by induction. We highlight the three items in the following example.

Example: Prove, by mathematical induction that $7 | 5^{2n} - 2^{5n}$ for all $n \geq 1$.

Proof: First introduce the name $A(n)$ for the statement $7 | 5^{2n} - 2^{5n}$ where $n \in \mathbb{N}$. Out task is to show that $\forall n \in \mathbb{N} : A(n)$.

1. Check that $A(1)$ is true. Is it? Does 7 divide $5^{2 \cdot 1} - 2^{5 \cdot 1}$? Well, $5^{2 \cdot 1} - 2^{5 \cdot 1} = 25 - 32 = -7 = 7 \cdot -1$ and this is divisible by 7 so, yes, $A(1)$ is true.
2. Prove the implication $A(p) \Rightarrow A(p+1)$ for every natural number p. We therefore assume $A(p)$, this is the *induction assumption* and it states that $7 | 5^{2p} - 2^{5p}$. This means that there exists a number q such that $5^{2p} - 2^{5p} = 7q$. With the support of this statement we wish to prove that $7 | 5^{2(p+1)} - 2^{5(p+1)}$. We therefore observe
$$5^{2(p+1)} - 2^{5(p+1)} = 5^{2p+2} - 2^{5p+5} = 25 \cdot 5^{2p} - 32 \cdot 2^{5p} = 25 \cdot 5^{2p} - 25 \cdot 2^{5p} - 7 \cdot 2^{5p}.$$
But this number can be written $25 \cdot (5^{2p} - 2^{5p}) - 7 \cdot 2^{5p}$, and, using the induction hypothesis which state that $5^{2p} - 2^{5p} = 7q$ for a natural number q, we have
$$5^{2(p+1)} - 2^{5(p+1)} = 25 \cdot 7q - 7 \cdot 2^{5p} = 7 \cdot (25q - 2^{5p})$$
which is clearly divisible by 7. Hence $7 | 5^{2(p+1)} - 2^{5(p+1)}$ which is $A(p+1)$. We have now proven the implication $A(p) \Rightarrow A(p+1)$ which completes step 2.
3. The two above steps and the principle of mathematical induction allows us to draw the conclusion that
$$A(1) \text{ is true } \Rightarrow A(2) \text{ is true } \Rightarrow \ldots \Rightarrow A(n) \text{ is true for all } n \geq 1$$
which completes the proof.

Strong Mathematical Induction

We will now study something called *strong* mathematical induction. It is totally equivalent to the ordinary type of mathematical induction that we have studied so far, but it is sometimes more useful.

Principle of Mathematical Induction, strong form: Given a statement $A(n)$ for every integer n. If

1. $A(n_0)$ is true for some integer n_0;
2. if $k > n_0$ is any integer and $A(p)$ is true for all integers p with $n_0 \leq p < k$, then we also have $A(k)$.

Then $A(n)$ is true for all integers greater than or equal to n_0, that is we have $\forall n \geq n_0 : A(n)$.

We will use the strong form of mathematical induction to prove that any natural number greater than 2 can be written as the product of primes. This was proven before too, but using the well-ordering principle of the natural numbers, and in fact, the well-ordering principle of the natural numbers is equivalent to the principle of mathematical induction.

Theorem: Every natural number greater than or equal to 2 is a product of primes.

Proof: We proceed by strong mathematical induction. We first introduce the predicate

$$A(n) \Leftrightarrow n \text{ is a product of primes.}$$

What we are to prove can then be expressed $\forall n \in \mathbb{N} : n \geq 2 \Rightarrow n$ is a product of primes. We now take the three steps necessary for mathematical induction:

1. The starting value is $n_0 = 2$ and $A(n_0) = A(2)$ is true since 2 itself is a prime, therefore it is also a product of primes (containing one factor, namely 2 itself.)
2. We now fix an arbitrary natural number $p \geq 2$ and assume that the statement holds for all natural numbers, $2, 3, \ldots, p-1$, that is we assume that $A(2), A(3), \ldots, A(p-1)$ all hold. Then either p is a prime itself, in which case the statement $A(p)$ holds (that p is a product of primes). The other possibility is that the number p is not a prime, in that case there exists natural numbers, $2 \leq a, b < p$ such that $p = a \cdot b$. But as both $A(a)$ and $A(b)$ hold, then both a, b must be products of primes, this in turn implies that $p = a \cdot b$ is a product of primes and we have then shown that $A(p)$ is true. We have now shown $A(2) \wedge A(3) \wedge \ldots \wedge A(p-1) \Rightarrow A(p)$ which is the second step in the proof.
3. The third step is to appeal to the principle of strong mathematical induction in which we conclude that as step 1 and 2 above proves we can deduce

$$A(2) \text{ true } \Rightarrow A(3) \text{ true } \Rightarrow \ldots \Rightarrow A(n) \text{ true for all } n \geq 2$$

which completes the proof. (The last step is often referred to as an "appeal to the principle of mathematical induction" and you MUST always include that step to have a complete proof.)

The proof is complete.

The best thing to do now is to produce as many proofs as you can based on mathematical induction. There are many such exercises in the book.

Recursively defined sequences

We will now use the principle of mathematical induction and congruences to study recursively defined sequences. A recursively defined sequence is a sequence of numbers (often integers) that are defined by specifying starting value(s) and then specifying how a new value depends on the previously defined values. We give an example:

Example: Let $a_1 = 1$ and $a_{n+1} = n \cdot a_n$, for $n \geq 1$. Then the value a_2 is defined to be $2 \cdot a_1 = 2 \cdot 1 = 2$. Similarly the value of a_3 is defined to be $3 \cdot a_2 = 3 \cdot 2 = 6$, and, continuing on, we find that $a_4 = 4 \cdot 3 \cdot \cdot 2 \cdot 1$ and so on, we have, generally

$$a_n = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1.$$

This number has important meaning in mathematics and is called the *factorial* and we denote it $n!$, and it is an example of something that can be recursively defined. Note that when we define something recursively it is very closely related to mathematical induction and indeed it is often very easy to prove something about a recursively defined sequence using mathematical induction. We take an example of this:

Example: For every $n \geq 4$, show that $n! \geq 2^n$.

Proof: We proceed by mathematical induction: introduce the name $A(n)$ for the statement that $n! \geq 2^n$. We wish to prove that $\forall n \geq 4 : A(n)$. We now take the three steps:

1. The statement $A(4)$ is true since $4! = 24 \geq 2^4 = 16$.
2. Now prove that $A(p) \Rightarrow A(p+1)$ for all $p \geq 4$ so we assume that $A(p)$ is true for a certain $p \geq 4$. With the support of this, we need to show that we also have $A(p+1)$. Study the statement $A(p+1)$ in detail: $A(p+1) \Leftrightarrow (p+1)! \geq 2^{p+1}$. Can this be true? Well, since $A(p) \Leftrightarrow p! \geq 2^p$ is true, we have that the left hand side is $(p+1)! = (p+1)p! \geq (p+1)2^p$, and since $p \geq 4$ we can further estimate this downwards so that it is greater than $2 \cdot 2^p = 2^{p+1}$. But in conclusion we have then found $(p+1)! \geq 2^{p+1}$ which is exactly the statement $A(p+1)$. Thus the induction step is successfully taken.
3. We now conventionally appeal to the principle of mathematical induction which gives that

$$A(1) \text{ is true } \Rightarrow A(2) \text{ is true } \Rightarrow \ldots \Rightarrow A(n) \text{ is true } \forall n \geq 4.$$

The proof is complete. (Question: is $A(p) \Rightarrow A(p+1)$ true for smaller p?)

It IS always important to take the third step in a proof based on mathematical induction since it is actually an appeal to a property of the natural numbers called the *axiom of induction* that can be formulated like this

$$\text{If } 1 \in S \subseteq \mathbb{N} \text{ and } \forall x \in S : x+1 \in S \text{ then } S = \mathbb{N}.$$

We will NOT regard a proof of based on mathematical induction to be complete without the third step.

We will now study two special types of sequences. We define both of these in the same definition:

Definition: A sequence of numbers $\{a_n\}_{n=1}^{\infty}$ is called an *arithmetic sequence* if there exists a constant d such that $a_{n+1} = a_n + d$ for all $n \in \mathbb{N}$. An sequence of numbers $\{a_n\}_{n=1}^{\infty}$ is called a *geometric sequence* if there exists a constant a such that $a_{n+1} = a \cdot a_n$ for all $n \in \mathbb{N}$. The number a is called the *quotient* of the geometric series. The values a_1 and b_1 are called the *starting values* of the sequences.

We will prove a theorem about them by induction and state another theorem and leave the proof as an exercise on induction proofs:

Theorem: Let $\{a_n\}_{n=1}^{\infty}$ be an arithmetic sequence and let $\{b_n\}_{n=1}^{\infty}$ be a geometric sequence. Then, for all $n \in \mathbb{N}$ we have $a_n = a_1 + (n-1) \cdot d$ and $b_n = b_1 \cdot b^{(n-1)}$, where b is the quotient of $\{b_n\}$.

Proof: This is a straight-forward proof by induction so we introduce the predicate

$$A(n) \Leftrightarrow a_n = a_1 + (n-1) \cdot d.$$

Our task is to show that $\forall n \in \mathbb{N} : A(n)$. We take the three steps of mathematical induction:

1. $A(1) \Leftrightarrow a_1 = a_1 + (1-1) \cdot d \Leftrightarrow a_1 = a_1$ which is obviously true.
2. We now proceed to show the implication $A(p) \Rightarrow A(p+1)$ and we therefore assume that $A(p)$ is true for a certain $p \in \mathbb{N}$, this is equivalent to $LHS_p = a_p = a_1 + (p-1) \cdot d = RHS_p$. Based on this we wish to prove that $A(p+1)$, that is we want to show that $LHS_{p+1} = RHS_{p+1}$. So we study
$$LHS_{p+1} = a_{p+1} = \{\textit{by the definition of an arithmetic sequence}\} = a_p + d = LHS_p + d$$
but this number is $RHS_p + d$, according to the induction assumption $A(p)$, so the whole expression is $a_1 + (p-1) \cdot d + d = a_1 + p \cdot d = a_1 + (p+1-1) \cdot d = RHS_{p+1}$ so that $LHS_{p+1} = RHS_{p+1}$ which is exactly the statement $A(p+1)$. We have therefore shown that the implication $A(p) \Rightarrow A(p+1)$ is always true for all $p \in \mathbb{N}$.
3. We now appeal to the principle of mathematical induction which gives that
$$A(1) \text{ is true } \Rightarrow A(2) \text{ is true } \Rightarrow \ldots \Rightarrow A(n) \text{ is true } \forall n \geq 1.$$

The proof is complete. Well not really, we have only done this for the arithmetic sequence, but the exact same procedure works for the geometric sequence too. You can formulate those details as an exercise.

We now state a theorem without proof leaving the details as a very good exercise on induction proofs:

Theorem: Let $\{a_n\}$ and $\{b_n\}$ be arithmetic and geometric sequences as in the definition above. Then

$$\sum_{k=1}^{n} a_k = n \cdot \frac{a_1 + a_n}{2} \quad \text{and} \quad \sum_{k=1}^{n} b_k = b_1 \frac{b^{n+1} - 1}{b - 1} \text{ if } b \neq 1, \text{ otherwise } \sum_{k=1}^{n} b_k = n \cdot b_1.$$

Solving recurrence relations

There is a special sequence of interest recursively defined by

$$a_0 = 1, a_1 = 1, a_{n+2} = a_{n+1} + a_n, \text{ for all } n \geq 0.$$

These numbers are called the *Fibonacci numbers* and they are given by $1, 1, 2, 3, 5, 8, 13, 21, \ldots$. They are a special example of a so-called recurrence relation. We will see how to solve any recurrence relation of the form

$$a_{n+2} = A \cdot a_{n+1} + B \cdot a_n, \text{ for all } n \geq 0$$

where we denote by a_0 and a_1 the starting values of this sequence. In fact it is solved in the same way as a differential equation, we first form the characteristic equation

$$\lambda^2 = A \cdot \lambda + B$$

and if this equation has the distinct roots r_1 and r_2, the solution to the recurrence relation is given by

$$a_n = C \cdot r_1^n + D \cdot r_2^n$$

where C, D are determined by A, B. If the characteristic equation $\lambda^2 = A\cdot\lambda + B$ has a double root r, the solution is given by $a_n = (C\cdot n + D)r^n$, where, again, C, D are determined by A, B. Again this can be formulated as a theorem and proven by mathematical induction, but we will study several examples instead. (Formulate this is as a theorem and do the proof as an exercise.)

Example: Find the sequence defined by $a_0 = 3, a_1 = 2, a_{n+2} = 3a_{n+1} - 2a_n$.

Solution: The characteristic equation is $\lambda^2 = 3\lambda - 2 \Leftrightarrow \lambda = 1 \vee \lambda = 2$, that is we have two distinct roots. Hence the sequence is given by $a_n = C\cdot 1^n + D\cdot 2^n$ and setting $n = 0$ and $n = 1$ gives the two equations $C + D = 3$ and $C + 2\cdot D = 2$ which is a linear system of equations which has the solution $C = 4$ and $D = -1$ so that the formula for the sequence reads $a_n = 4 - 2^n$.

If the characteristic equation has a double root, r, the solution will read $a_n = (C\cdot n + D)r^n$. We illustrate this with an example:

Example: Solve the reccurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$, $n \geq 0$, where $a_0 = 1$ and $a_1 = 4$.

Solution: The characteristic equation reads $\lambda^2 = 4\lambda - 4 \Leftrightarrow (\lambda - 2)^2 = 0$ which has the double root $\lambda = 2$. Hence the solution to the recurrence relation is given by $a_n = (C\cdot n + D)2^n$ and setting $n = 0$ and $n = 1$ gives the equations $(C\cdot 0 + D)2^0 = a_0 = 1$ and $(C\cdot 1 + D)2^1 = 4$ which can be written

$$D = 1 \wedge C + D = 2 \Leftrightarrow C = D = 1$$

so the solution to the recurrence relation is $a_n = (C\cdot n + D)2^n = (n + 1)2^n$.

Solving a homogenous (=0) linear recurrence relation with constant coefficients of the second degree is easy. It will be an expression of the form $a_n = C\cdot r_1^n + D\cdot r_2^n$ or $a_n = (C\cdot n + D)r^n$ dependeing on whether we have two distinct roots or if we have a double root. It is just a matter of forming the characteristic equation and determining the constants by using the starting values. We conclude this part of the theory by studying a classic example of a recurrence relation and after that we will show how to solve non-homogenous linear recurrence relations with constant coefficients.

Example: Define the sequence of integers $\{a_n\}_{n=0}^{\infty}$ by $a_0 = 1, a_1 = 1$ and $a_{n+2} = a_{n+1} + a_n$.

Solution: The characteristic equation is $\lambda^2 = \lambda + 1$ which has the solutions $\lambda = \frac{1\pm\sqrt{5}}{2}$. These are two distinct roots so the solution has the general form

$$a_n = C\cdot\left(\frac{1+\sqrt{5}}{2}\right)^n + D\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Putting $n = 0, n = 1$ gives the equations $C + D = 1$ and $C\cdot(\frac{1+\sqrt{5}}{2}) + D\cdot(\frac{1-\sqrt{5}}{2}) = 1$ which has the solutions $C = \frac{1}{\sqrt{5}}\frac{1+\sqrt{5}}{2}$, $D = \frac{1}{\sqrt{5}}\frac{1-\sqrt{5}}{2}$, so that the general form of the solution reads $a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$. This sequence of numbers is called the *Fibonacci* numbers, they are $1, 1, 2, 3, 5, 8, 13, 21, \ldots$. It is a bit curious that the complicated expression that defines a_n that we have found always turn out to be natural numbers.

The non-homogenous case. We now turn to the case where the recurrence relation is not homogenous. In this theory we proceed exactly as with differential equations, we study this through example.

Example: Solve the linear recurrence relation $a_{n+2} = 5a_{n+1} - 6a_n + 4n$ for various starting values a_0, a_1.

Solution: The equation is not homogenous, therefore we need to find a so-called *particular solution*, that is we need to find one sequence $\{a_n^p\}_{n=0}^{\infty}$ such that $a_{n+2}^p = 5a_{n+1}^p - 6a_n^p + 4n$ holds. The general solution to $a_{n+2} = 5a_{n+1} - 6a_n + 4n$ will then be the solution to the homogenous equation $a_{n+2} = 5a_{n+1} - 6a_n$ which will be of the form $a_n = C\cdot 2^n + D\cdot 3^n$, as it has the characteristic equation $\lambda^2 - 5\lambda + 6 = 0$ which has solutions $r_1 = 2, r_2 = 3$. So we proceed to find a so-called particular solution. This sometimes requires some ingenuity. As the expression $4n$ is a polynomial in n we try to set a particular solution as a polynomial, assume $a_n^p = qn+r$, where q, r are real numbers. Then $a_{n+2}^p = 5a_{n+1}^p - 6a_n^p + 4n \Leftrightarrow q(n+2)+r = 5q(n+1)+5r-6qn-6r+4n$. We continue to work with equivalences and find that this is equivalent to $qn+2q+r = 5qn+5q+5r-6qn-6r+4n \Leftrightarrow 4n - 2qn + 3q - 2r = 0 \Leftrightarrow (4 - 2q)n + (3q - 2r) = 0$. This is a polynomial in n which is supposed to always be 0, and it is 0 if and only if $4 - 2q = 0$ and $3q - 2r = 0$ which in turn is equivalent to $q = 2, r = 3$. (Observe that it si crucial that we wrok with equivalences all the way through!) So we have found a particular solution being $a_n^p = 2n + 3$ and then the general form of a solution to $a_{n+2} = 5a_{n+1} - 6a_n + 4n$ will then be $a_n = C \cdot 2^n + D \cdot 3^n + 2n + 3$ where C, D are determined by the starting values of the sequence. For example if $a_0 = 1, a_1 = 2$, then we obtain the equations $C + D + 3 = 1$ and $2C + 3D + 2 + 3 = 2$ which have the solution $C, D = -3, 1$ giving the solution $a_n = 3^n - 3 \cdot 2^n + 2n + 3$. Other starting values yields other equations for C, D and hence give rise to other solutions.