Theorem 7.3 In a bipartite graph G with $\delta > 0$, the number of vertices in a maximum independent set is equal to the number of edges in a minimum edge covering.

Proof Let G be a bipartite graph with $\delta > 0$. By corollary 7.1 and theorem 7.2, we have

$$\alpha + \beta = \alpha' + \beta'$$

and, since G is bipartite, it follows from theorem 5.3 that $\alpha' = \beta$. Thus $\alpha = \beta'$ □

Even though the concept of an independent set is analogous to that of a matching, there exists no theory of independent sets comparable to the theory of matchings presented in chapter 5; for example, no good algorithm for finding a maximum independent set in a graph is known. However, there are two interesting theorems that relate the number of vertices in a maximum independent set of a graph to various other parameters of the graph. These theorems will be discussed in sections 7.2 and 7.3.

Exercises

7.1.1 (*a*) Show that G is bipartite if and only if $\alpha(H) \geq \frac{1}{2}\nu(H)$ for every subgraph H of G.
(*b*) Show that G is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph H of G such that $\delta(H) > 0$.

7.1.2 A graph is *α-critical* if $\alpha(G-e) > \alpha(G)$ for all $e \in E$. Show that a connected α-critical graph has no cut vertices.

7.1.3 A graph G is *β-critical* if $\beta(G-e) < \beta(G)$ for all $e \in E$. Show that
(*a*) a connected β-critical graph has no cut vertices;
(*b*)* if G is connected, then $\beta \leq \frac{1}{2}(\varepsilon + 1)$.

7.2 RAMSEY'S THEOREM

In this section we deal only with simple graphs. A *clique* of a simple graph G is a subset S of V such that $G[S]$ is complete. Clearly, S is a clique of G if and only if S is an independent set of G^c, and so the two concepts are complementary.

If G has no large cliques, then one might expect G to have a large independent set. That this is indeed the case was first proved by Ramsey (1930). He showed that, given any positive integers k and l, there exists a smallest integer $r(k, l)$ such that every graph on $r(k, l)$ vertices contains either a clique of k vertices or an independent set of l vertices. For example, it is easy to see that

$$r(1, l) = r(k, 1) = 1 \tag{7.5}$$

and

$$r(2, l) = l, \qquad r(k, 2) = k \tag{7.6}$$

The numbers $r(k, l)$ are known as the *Ramsey numbers*. The following theorem on Ramsey numbers is due to Erdös and Szekeres (1935) and Greenwood and Gleason (1955).

Theorem 7.4 For any two integers $k \geq 2$ and $l \geq 2$

$$r(k, l) \leq r(k, l-1) + r(k-1, l) \tag{7.7}$$

Furthermore, if $r(k, l-1)$ and $r(k-1, l)$ are both even, then strict inequality holds in (7.7).

Proof Let G be a graph on $r(k, l-1) + r(k-1, l)$ vertices, and let $v \in V$. We distinguish two cases:

(i) v is nonadjacent to a set S of at least $r(k, l-1)$ vertices, or
(ii) v is adjacent to a set T of at least $r(k-1, l)$ vertices.

Note that either case (i) or case (ii) must hold because the number of vertices to which v is nonadjacent plus the number of vertices to which v is adjacent is equal to $r(k, l-1) + r(k-1, l) - 1$.

In case (i), $G[S]$ contains either a clique of k vertices or an independent set of $l-1$ vertices, and therefore $G[S \cup \{v\}]$ contains either a clique of k vertices or an independent set of l vertices. Similarly, in case (ii), $G[T \cup \{v\}]$ contains either a clique of k vertices or an independent set of l vertices. Since one of case (i) and case (ii) must hold, it follows that G contains either a clique of k vertices or an independent set of l vertices. This proves (7.7).

Now suppose that $r(k, l-1)$ and $r(k-1, l)$ are both even, and let G be a graph on $r(k, l-1) + r(k-1, l) - 1$ vertices. Since G has an odd number of vertices, it follows from corollary 1.1 that some vertex v is of even degree; in particular, v cannot be adjacent to precisely $r(k-1, l) - 1$ vertices. Consequently, either case (i) or case (ii) above holds, and therefore G contains either a clique of k vertices or an independent set of l vertices. Thus

$$r(k, l) \leq r(k, l-1) + r(k-1, l) - 1$$

as stated □

The determination of the Ramsey numbers in general is a very difficult unsolved problem. Lower bounds can be obtained by the construction of suitable graphs. Consider, for example, the four graphs in figure 7.2.

The 5-cycle (figure 7.2*a*) contains no clique of three vertices and no independent set of three vertices. It shows, therefore, that

$$r(3, 3) \geq 6 \tag{7.8}$$

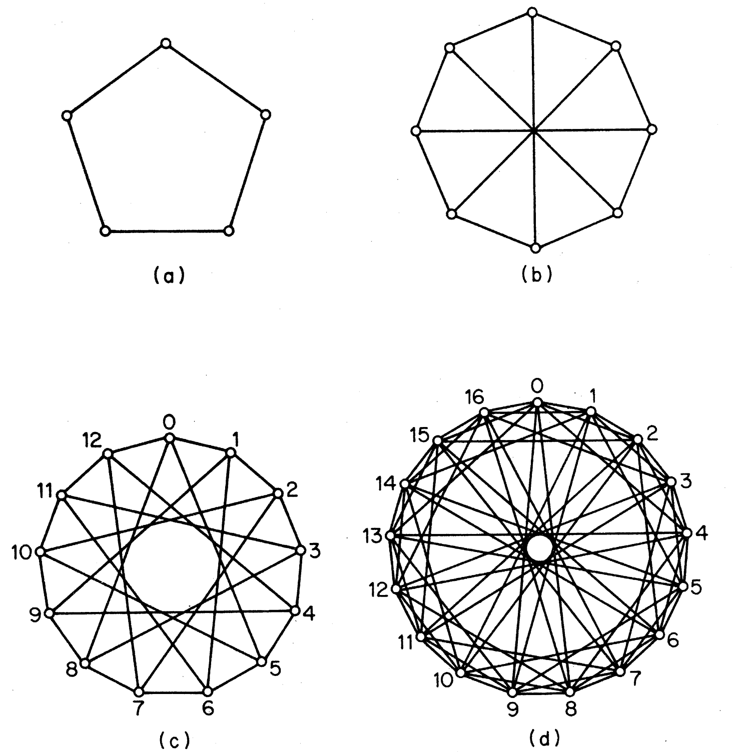

Figure 7.2. (*a*) A (3,3)-Ramsey graph; (*b*) a (3,4)-Ramsey graph; (c) a (3,5)-Ramsey graph; (*d*) a (4,4)-Ramsey graph

The graph of figure 7.2*b* contains no clique of three vertices and no independent set of four vertices. Hence

$$r(3, 4) \geq 9 \tag{7.9}$$

Similarly, the graph of figure 7.2*c* shows that

$$r(3, 5) \geq 14 \tag{7.10}$$

and the graph of figure 7.2*d* yields

$$r(4, 4) \geq 18 \tag{7.11}$$

With the aid of theorem 7.4 and equations (7.6) we can now show that equality in fact holds in (7.8), (7.9), (7.10) and (7.11). Firstly, by (7.7) and (7.6)

$$r(3, 3) \leq r(3, 2) + r(2, 3) = 6$$

and therefore, using (7.8), we have $r(3, 3) = 6$. Noting that $r(3, 3)$ and $r(2, 4)$ are both even, we apply theorem 7.4 and (7.6) to obtain

$$r(3, 4) \leq r(3, 3) + r(2, 4) - 1 = 9$$

With (7.9) this gives $r(3, 4) = 9$. Now we again apply (7.7) and (7.6) to obtain

$$r(3, 5) \leq r(3, 4) + r(2, 5) = 14$$

and

$$r(4, 4) \leq r(4, 3) + r(3, 4) = 18$$

which, together with (7.10) and (7.11), respectively, yield $r(3, 5) = 14$ and $r(4, 4) = 18$.

The following table shows all Ramsey numbers $r(k, l)$ known to date.

$k \backslash l$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	9	14	18	23
4	1	4	9	18			

A (k, l)-*Ramsey graph* is a graph on $r(k, l) - 1$ vertices that contains neither a clique of k vertices nor an independent set of l vertices. By definition of $r(k, l)$ such graphs exist for all $k \geq 2$ and $l \geq 2$. Ramsey graphs often seem to possess interesting structures. All of the graphs in figure 7.2 are Ramsey graphs; the last two can be obtained from finite fields in the following way. We get the (3, 5)-Ramsey graph by regarding the thirteen vertices as elements of the field of integers modulo 13, and joining two vertices by an edge if their difference is a cubic residue of 13 (either 1, 5, 8 or 12); the (4, 4)-Ramsey graph is obtained by regarding the vertices as elements of the field of integers modulo 17, and joining two vertices if their difference is a quadratic residue of 17 (either 1, 2, 4, 8, 9, 13, 15 or 16). It has been conjectured that the (k, k)-Ramsey graphs are always self-complementary (that is, isomorphic to their complements); this is true for $k = 2$, 3 and 4.

In general, theorem 7.4 yields the following upper bound for $r(k, l)$.

Theorem 7.5 $r(k, l) \leq \binom{k+l-2}{k-1}$

Proof By induction on $k + l$. Using (7.5) and (7.6) we see that the theorem holds when $k + l \leq 5$. Let m and n be positive integers, and assume that the theorem is valid for all positive integers k and l such that

$5 \le k+l < m+n$. Then, by theorem 7.4 and the induction hypothesis

$$r(m,n) \le r(m,n-1) + r(m-1,n)$$
$$\le \binom{m+n-3}{m-1} + \binom{m+n-3}{m-2} = \binom{m+n-2}{m-1}$$

Thus the theorem holds for all values of k and l □

A lower bound for $r(k,k)$ is given in the next theorem. It is obtained by means of a powerful technique known as the *probabilistic method* (see Erdös and Spencer, 1974). The probabilistic method is essentially a crude counting argument. Although nonconstructive, it can often be applied to assert the existence of a graph with certain specified properties.

Theorem 7.6 (Erdös, 1947) $r(k,k) \ge 2^{k/2}$

Proof. Since $r(1,1)=1$ and $r(2,2)=2$, we may assume that $k \ge 3$. Denote by $\mathscr{G}_n$ the set of simple graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$, and by $\mathscr{G}_n^k$ the set of those graphs in $\mathscr{G}_n$ that have a clique of k vertices. Clearly

$$|\mathscr{G}_n| = 2^{\binom{n}{2}} \tag{7.12}$$

since each subset of the $\binom{n}{2}$ possible edges v_iv_j determines a graph in $\mathscr{G}_n$. Similarly, the number of graphs in $\mathscr{G}_n$ having a particular set of k vertices as a clique is $2^{\binom{n}{2}-\binom{k}{2}}$. Since there are $\binom{n}{k}$ distinct k-element subsets of $\{v_1, v_2, \ldots, v_n\}$, we have

$$|\mathscr{G}_n^k| \le \binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}} \tag{7.13}$$

By (7.12) and (7.13)

$$\frac{|\mathscr{G}_n^k|}{|\mathscr{G}_n|} \le \binom{n}{k} 2^{-\binom{k}{2}} < \frac{n^k 2^{-\binom{k}{2}}}{k!} \tag{7.14}$$

Suppose, now, that $n < 2^{k/2}$. From (7.14) it follows that

$$\frac{|\mathscr{G}_n^k|}{|\mathscr{G}_n|} < \frac{2^{k^2/2}2^{-\binom{k}{2}}}{k!} = \frac{2^{k/2}}{k!} < \tfrac{1}{2}$$

Therefore, fewer than half of the graphs in $\mathscr{G}_n$ contain a clique of k vertices. Also, because $\mathscr{G}_n = \{G \mid G^c \in \mathscr{G}_n\}$, fewer than half of the graphs in $\mathscr{G}_n$ contain an independent set of k vertices. Hence some graph in $\mathscr{G}_n$ contains neither a clique of k vertices nor an independent set of k vertices. Because this holds for any $n < 2^{k/2}$, we have $r(k,k) \ge 2^{k/2}$ □

From theorem 7.6 we can immediately deduce a lower bound for $r(k,l)$.

Corollary 7.6 If $m = \min\{k, l\}$, then $r(k,l) \ge 2^{m/2}$

All known lower bounds for $r(k,l)$ obtained by constructive arguments are much weaker than that given in corollary 7.6; the best is due to Abbott (1972), who shows that $r(2^n+1, 2^n+1) \ge 5^n+1$ (exercise 7.2.4).

The Ramsey numbers $r(k,l)$ are sometimes defined in a slightly different way from that given at the beginning of this section. One easily sees that $r(k,l)$ can be thought of as the smallest integer n such that every 2-edge colouring (E_1, E_2) of K_n contains either a complete subgraph on k vertices, all of whose edges are in colour 1, or a complete subgraph on l vertices, all of whose edges are in colour 2. Expressed in this form, the Ramsey numbers have a natural generalisation. We define $r(k_1, k_2, \ldots, k_m)$ to be the smallest integer n such that every m-edge colouring $(E_1, E_2, \ldots, E_m)$ of K_n contains, for some i, a complete subgraph on k_i vertices, all of whose edges are in colour i.

The following theorem and corollary generalise (7.7) and theorem 7.5, and can be proved in a similar manner. They are left as an exercise (7.2.2).

Theorem 7.7 $r(k_1, k_2, \ldots, k_m) \le r(k_1-1, k_2, \ldots, k_m) + r(k_1, k_2-1, \ldots, k_m) + \ldots + r(k_1, k_2, \ldots, k_m-1) - m + 2$

Corollary 7.7 $r(k_1+1, k_2+1, \ldots, k_m+1) \le \dfrac{(k_1+k_2+\ldots+k_m)!}{k_1!\,k_2!\ldots k_m!}$

Exercises

7.2.1 Show that, for all k and l, $r(k,l) = r(l,k)$.

7.2.2 Prove theorem 7.7 and corollary 7.7.

7.2.3 Let r_n denote the Ramsey number $r(k_1, k_2, \ldots, k_n)$ with $k_i = 3$ for all i.

(*a*) Show that $r_n \le n(r_{n-1}-1)+2$.
(*b*) Noting that $r_2 = 6$, use (*a*) to show that $r_n \le [n!\,e]+1$.
(*c*) Deduce that $r_3 \le 17$.
(Greenwood and Gleason, 1955 have shown that $r_3 = 17$.)

7.2.4 The *composition* of simple graphs G and H is the simple graph $G[H]$ with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

(*a*) Show that $\alpha(G[H]) \le \alpha(G)\alpha(H)$.
(*b*) Using (*a*), show that

$$r(kl+1, kl+1) - 1 \ge (r(k+1, k+1)-1) \times (r(l+1, l+1)-1)$$

(*c*) Deduce that $r(2^n+1, 2^n+1) \ge 5^n+1$ for all $n \ge 0$.

(H. L. Abbott)

2. Classical Two-Color Ramsey Numbers

2.1. Values and bounds for $R(k,l)$, $k \le 10$, $l \le 15$

k \ l	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 42	47 50	52 59	59 68	66 77	73 87
4		18	25	36 41	49 61	58 84	73 115	92 149	98 191	128 238	133 291	141 349	153 417
5			43 49	58 87	80 143	101 216	126 316	144 442	171 633	191 848	213 1138	239 1461	265 1878
6				102 165	113 298	132 495	169 780	179 1171	253 1804	263 2566	317 3703	5033	401 6911
7					205 540	217 1031	241 1713	289 2826	405 4553	417 6954	511 10578	15263	22112
8						282 1870	317 3583	6090	10630	16944	817 27485	41525	861 63609
9							565 6588	581 12677	22325	38832	64864		
10								798 23556	45881	81123			1265

Table I. Known nontrivial values and bounds for two color Ramsey numbers $R(k,l) = R(k,l;2)$.

k \ l	4	5	6	7	8	9	10	11	12	13	14	15
3	GG	GG	Kéry	Ka2 GrY	GR MZ	Ka2 GR	Ex5 GoR1	Ex20 GoR1	Ex12 Les	Piw1 GoR1	Ex8 GoR1	WW GoR1
4	GG	Ka1 MR4	Ex19 MR5	Ex3 Mac	Ex20 Mac	Ex16 Mac	HaKr1 Mac	Ex17 Spe4	SLL Spe4	2.3.e Spe4	XXR Spe4	XXR Spe4
5		Ex4 MR5	Ex9 HZ1	CaET Spe4	HaKr1 Spe4	Ex17 Mac	Ex17 Mac	Gerb HW+	Gerb HW+	Gerb HW+	Gerb HW+	Ex16 HW+
6			Ka1 Mac	Ex16 Mac	XSR2 Mac	XXER Mac	Ex16 Mac	XXR HW+	XSR2 HW+	XXER HW+	HW+	2.3.h HW+
7				She2 Mac	XSR2 Mac	XSR2 HZ1	2.3.h Mac	XXER HW+	XSR2 HW+	XXR HW+	HW+	HW+
8					BR Mac	XXER Ea1	HZ1	HW+	HW+	XXER HW+	HW+	2.3.h HW+
9						She2 ShZ1	XSR2 Ea1	HW+	HW+	HW+		
10							She2 Shi2	HW+	HW+			2.3.h

References for Table I;
HW+ abbreviates HWSYZH, as enhanced by Boza [Boza5], see 2.1.m.