Mid-term exam 1 for HF1013, Discrete Mathematics:TEN1. Solutions.

Allowed aids: A sheet of hand-written notes, notes on **one** side only. (No computer print-outs!) And one pocket calculator.

Problems

Give complete and correct solutions, write legibly. Motivate each logic step fully. No points for answers only. Each problem is worth 3 points. A total of 4 points is needed to pass the exam.

1. Let A, B, C be arbitrary sets and consider the symmetric difference between two sets:
$$A \triangle B = (A - B) \cup (B - A).$$
Prove that every element in the set $A \triangle B \triangle C$ lies in exactly 1 or 3 of the sets A, B, C.

Låt A, B, C vara godtyckliga mängder och betrakta den symmetriska differensen mellan två mängder:
$$A \triangle B = (A - B) \cup (B - A).$$
Bevisa att varje element som ligger i mängden $A \triangle B \triangle C$ ligger i exakt 1 eller 3 av mängderna A, B, C.

Sol: Another way of writing the difference between two sets A, B is $A - B = A \cap B^c$. Then, by definition
$$A \triangle B \triangle C = ((A \triangle B) - C) \cup (C - (A \triangle B)) =$$
$$(((A - B) \cup (B - A)) - C) \cup (C - ((A - B) \cup (B - A))) =$$
$$(((A \cap B^c) \cup (B \cap A^c)) \cap C^c) \cup (C \cap ((A \cap B^c) \cup (B \cap A^c))^c) = M_1 \cup M_2.$$
Then, by set laws (including DeMorgan's) we get
$$M_1 = ((A \cap B^c) \cup (B \cap A^c)) \cap C^c = (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) =$$
$$(A \cap (B \cup C)^c) \cup (B \cap (A \cup C)^c) = (A - (B \cup C)) \cup (B - (A \cup C))$$
and
$$M_2 = C \cap ((A \cap B^c) \cup (B \cap A^c))^c = C \cap ((A \cap B^c)^c \cap (B \cap A^c)^c) = C \cap ((A^c \cup B) \cap (B^c \cup A)).$$
But by the distributive law we have $(A^c \cup B) \cap (B^c \cup A) = (A^c \cap B^c) \cup (A \cap A^c) \cup (B \cap B^c) \cup (A \cap B)$, and since $A \cap A^c = B \cap B^c = \emptyset$ we get
$$M_2 = C \cap ((A^c \cap B^c) \cup (A \cap B)) = (C - (A \cup B)) \cup (A \cap B \cap C)$$
so that
$$A \triangle B \triangle C = M_1 \cup M_2 = (A - (B \cup C)) \cup (B - (A \cup C)) \cup (C - (A \cup B)) \cup (A \cap B \cap C).$$
These four sets are all disjoint and hence every element in $A \triangle B \triangle C$ lies in precisely one of them. But the first three sets are the elements that lie in exactly one of the sets A, B, C that is in precisely 1 of them, and the fourth set is the set of all elements that lie in all of the sets, that is in 3 of them. The proof is complete.

This problem can also be solved by using Venn diagrams.

2. Use mathematical induction to show that for any integer $n \geq 1$,
$$4 | 3^{2n-1} + 1.$$

Använd matematisk induktion för att visa att så fort $n \geq 1$ är ett heltal så gäller
$$4 | 3^{2n-1} + 1.$$

Sol: Introduce the predicate $A(n) \Leftrightarrow 4|3^{2n-1}+1$. We need to prove $\forall n \in \mathbb{N} : A(n)$. We do this by mathematical induction which has three steps.

1. Check that $A(1)$ is true. Is it? Well, $A(1)$ claims that $4|3^(2\cdot 1-1)+1$ but $3^(2\cdot 1-1)+1 = 3^1+1 = 4$ and we have indeed $4|4$ so $A(1)$ is true.
2. Now prove that the implication $A(p) \Rightarrow A(p+1)$ is true for all $p \in \mathbb{N}$. Accordingly assume that $A(p)$ is true for a natural number p. This means that $4|3^{2p-1}+1$ which means that there exists an integer q such that $3^{2p-1}+1 = 4q$. From this we need to show that $A(p+1)$ is true that is we need to show that $4|3^{2(p+1)-1}+1$. So study $3^{2(p+1)-1}+1$ and try to see that is it divisible by 4:
$$3^{2(p+1)-1}+1 = 3^{2p+2-1}-1 = 3^2\cdot 3^{2p-1}+1 = 8\cdot 3^{2p-1}+3^{2p-1}+1. \tag{1}$$
This last calculation is motivated by that we need to use the induction assumption somehow, that is, we need to use the fact that $3^{2p-1}+1 = 4q$ for some q. Writing the expression in the above manner (1) enables us to see that
$$3^{2(p+1)-1}+1 = 8\cdot 3^{2p-1}+4\cdot q = 4\cdot(2\cdot 3^{2p-1}+q)$$
which is obviously divisible by 4. Hence we have show that $A(p+1)$ is true following from the truth of $A(p)$, that is the implication $A(p) \Rightarrow A(p+1)$ is true.
3. We get $A(1)$ true $\Rightarrow$ $A(2)$ true $\Rightarrow \ldots \forall n \in N : A(n)$ thanks to the principle of mathematical induction and the steps 1 and 2.