The Obstacle Problem.

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1 Introduction.

The point of this last part of the course is to give you some exposure to advanced theory for partial differential equations. Modern theory of PDE is based on a variety of techniques from advanced analysis - in particular measure theory, Sobolev spaces and functional analysis. This means that normally one need to take at least three advanced courses in addition to an advanced PDE course before one can begin to comprehend modern PDE theory. This is unfortunate since, as so often in mathematics, many of the ideas are elementary. These notes are an attempt to introduce some modern PDE theory in an as gentle way as possible.

We will use the obstacle problem as a model problem for modern PDE theory. This choice is rather arbitrary, but there are some good reasons to discuss the obstacle problem:

- To show existence of solutions we will have a reason to introduce calculus of variations which might be the most important topic covered in these notes.
- There are no *"interesting"* C^2 solutions to the obstacle problem. This means that we need to relax the concept of solution in order to treat the equation. Since the 1950s relaxation of the concept of solution has been one of the standard tools in proving existence of solutions.
- The obstacle problem is a non-linear PDE - which makes it very interesting. But it is not too non-linear which would make it to difficult to talk about in three weeks.
- The obstacle problem is strongly related to the Laplace equation which means that we may use much of the theory that you have learned in previous parts of the course.

There are some problems in dealing with the obstacle problem. In particular, one cannot do any theory for the obstacle problem without using Sobolev spaces.[1] We will not develop Sobolev space theory in these notes. We will however provide some justifications, mostly in the one dimensional case, for the Sobolev space results that we will use in an appendix to the section on calculus of variations. Hopefully, you will gain enough intuitive knowledge of Sobolev spaces from the appendix in order to accept the proofs in the main body of the text.

Notation: We will use the letter D to denote a domain in $\mathbb{R}^n$ (n will always denote the space dimension) - that is D is an open set. Throughout these notes D will be bounded and connected. The topological boundary of D will be denoted ∂D and the outward pointing normal of D will be denoted ν. An open ball of radius r with center x^0 will be denoted $B_r(x^0)$. The upper half-ball will be denoted $B_r^+(x^0) = \{x \in B_r(x^0);\ x_n > 0\}$. By $\text{diam}(D)$ we mean the diameter of the set D - that is by definition the diameter of the smallest ball that contains the set D.

We will denote points in space by $x = (x_1, x_2, ...x_n)$, $y = (y_1, y_2, ..., y_n)$, $z = (z_1, ..., z_n)$ et.c. We often think of these vectors to be variables. Fixed points are often denoted by a superscript x^0, y^0 et.c. At times we will use a prime x' to denote the first $n-1$ components in a vector: $x' = (x_1, x_2, ..., x_{n-1})$. In a slight abuse of notation we will at times interpret x' as a vector in $\mathbb{R}^n$ with n:th component equal to zero $x' = (x_1, x_2, ..., x_{n-1}, 0)$ and at times we will also write $(x', t) = (x_1, x_2, .., x_{n-1}, t)$; in particular $x = (x', x_n)$. We believe that it it will be clear from context what we intend.

We will use several function spaces in these notes. By $C(D)$ we mean all the continuous functions in the domain D, by $C^{0,\alpha}(D)$ we mean the space of all continuous functions, $u(x)$, on D such that $|u(x) - u(y)| \le C|x - y|^\alpha$ equipped with the norm $\|u\|_{C^{0,\alpha}(D)} = \sup_{x \in D} |u(x)| + \sup_{x,y \in D} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$. If $\alpha \in (0,1)$

[1] Or viscosity solutions which we will not mention at all in these notes.

we often write $C^\alpha(D) = C^{0,\alpha}(D)$. Similarly we write $C^{k,\alpha}(D)$ for all k times continuously differentiable functions $u(x)$ defined on D such that all k :th order derivatives of $u(x)$ belong to $C^\alpha(D)$: that is $D^k u(x) \in C^\alpha(D)$. We will allow α to be zero and then just write $C^{k,\alpha}(D) = C^k(D)$.

We will use the spaces $L^2(D)$ for all integrable functions on D such that $\|u\|^2_{L^2(D)} = \int_D |u|^2 dx < \infty$. We will also use the Sobolev space $W^{1,2}(D)$ for all functions $u(x)$ defined on D such that both $u(x)$ and $\nabla u(x)$ are integrable and $\|u\|^2_{W^{1,2}(D)} = \|u\|^2_{L^2(D)} + \|\nabla u\|^2_{L^2(D)} < \infty$. Similarly we will use $W^{k,2}(D)$ for the space of functions such that all derivatives of order up to k belong to $L^2(D)$.

We will use a sub-script $C_c(D)$, $C_c^2(D)$ et.c. to denote the functions $u \in C_c(D)$, $u \in C_c^2(D)$ et.c with compact support. And $W^{k,2}(D)$ for functions that are identically equal to zero (in the trace sense) on ∂D.

We will use ∇ for the gradient operator $\nabla u(x) = \left(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, ..., \frac{\partial u(x)}{\partial x_n}\right)$ and Δ for the Laplace operator $\Delta u(x) = \sum_{j=1}^n \frac{\partial^2 u(x)}{\partial x_j^2}$.

2 The Calculus of Variations.

The calculus of variations consists in finding, and describing properties, of functions that minimize some energy. To be specific we look for a function $u(x)$ that minimizes the following energy

$$J(u) =_{\mathrm{df}} \int_D F(\nabla u(x))dx \tag{1}$$

among all functions in[2]

$$K = \{u \in W^{1,2}(D),\ u(x) = f(x) \text{ on } \partial D\}. \tag{2}$$

In physics the energy is usually some combination of several energies, for instance potential and kinetic energy. Calculus of variations is very important for applications. In these notes we will be interested in the mathematical theory.

The main problem in the calculus of variations is to show existence of minimizers to the minimization problem (1) in the set K. It is easy to construct examples of functionals for which no minimizers exists. The easiest example of a minimization problem for which no minima exists is for discontinuous functions defined on $\mathbb{R}$, which has nothing to do with Sobolev spaces.

[2]Here we use the notation $W^{1,2}(D)$ which is the set of all functions such that $\int_D (|\nabla u|^2 + |u|^2) < \infty$. This is a Sobolev space. We refer to the appendix to this section for more details on Sobolev spaces.

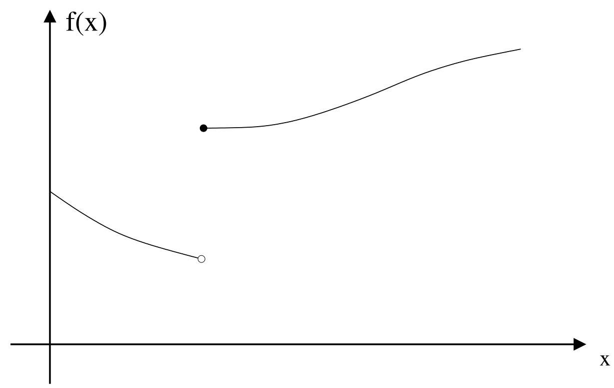


Figure 1: *A one dimensional example where the minimum does not exist.*

The example in figure 1 clearly shows that we need some assumptions in order to assure that minimizers for a given minimization problem exist. In order to understand the existence properties for the problem described by (1)-(2) we begin by stating a simple Theorem that we understand on minimization of functions in $\mathbb{R}^n$:

Theorem 2.1. *If $f(x)$ is a lower semi-continuous*[3] *function on a closed and bounded set $K \subset R^n$ and $f(x) > -\infty$ then $f(x)$ achieves its minimum in K.*

Proof: The proof is, as we already know, done in several simple steps.

1. $V_f = \{f(x);\ x \in K\}$ is bounded from below which implies, by the completeness property of the real numbers, that $\inf_{x\in K} f(x)$ exists.

2. We may thus find $x^j \in K$ s.t. $\lim_{j\to\infty} f(x^j) = \inf_{x\in K} f(x)$.

3. Since K is a compact set in $\mathbb{R}^n$ the Bolzano-Weierstrass Theorem implies that there exists a convergent sub-sequence of x^j which we will denote $x^{j_k} \to x^0 \in K$.

4. Lower semi-continuity of f implies that
$$f(x^0) \le \lim_{k\to\infty} f(x^{j_k}) = \inf_{x\in K} f(x).$$
Clearly by the definition of infimum $f(x_0) \ge \inf_{x\in K} f(x)$. It follows that $f(x_0) = \inf_{x\in K} f(x)$; thus $f(x)$ achieves its minimum in x^0.

□

We would like to replicate this theorem in the more complicated setting of the minimization of the functional (1) in the set (2). The main difference for the minimization problem ((1)-(2)):

$$\text{minimize } J(u) = \int_D F(\nabla u)dx \quad u \in K = \{W^{1,p}(D);\ u = f \text{ on } \partial D\}.$$

[3]Remember that $f(x)$ is lower semi-continuous if $f(x_0) = \liminf_{x\to x_0} f(x)$

is that K is no longer finite dimensional which implies that we no longer have a Bolzano-Weierstrass compactness Theorem.

But we may almost replicate the strategy of Theorem 2.1

1. To show that the minimum is a well defined number we just need to assume that $F(\nabla u) \geq -C$. That would imply that $J(u) \geq -C \int_D dx = -C|D|$; then the existence of a minimum of the functional exists by the completeness property of the real numbers. Notice that this does not imply that there exists a function $u \in K$ such that $J(u) = \inf_{u \in K} J(u)$.

2. By the property that an infimum exists we can find a sequence $u^j \in K$ s.t. $\lim_{j \to \infty} J(u^j) = \inf_{u \in K} J(u)$.

3. The space $W^{1,2}$ is weakly compact[4] so if

$$\|u^j\|_{W^{1,2}(D)} \leq C \tag{3}$$

then $u^j \rightharpoonup u^0 \in W^{1,2}(D)$.

In order to assure (3) we need to assume that the functional is **(Coercive)**:

$$J(u) \to \infty \text{ as } \|u\|_{W^{1,2}}(D) \to \infty. \tag{4}$$

Clearly (4) implies that $\|u^j\|_{W^{1,2}(D)}$ is bounded if $J(u^j)$ is bounded, which it certainly is if $J(u^j) \to \inf_{u \in K} J(u) \in \mathbb{R}$.

4. We need to show that $J(u)$ is lower semi-continuous with respect to weak convergence in $W^{1,2}(D)$.

In the above strategy there is no real problem with the first three points. We can clearly decide whether the first and third points holds if we have an explicit functional $J(u) = \int_D F(\nabla u) dx$ - at least we can easily imagine classes of functionals where the first and third point holds. The second point is just a simple fact that follows from the definition of the infimum.

The fourth point needs some further comment. In general, it is not meaningful to have theorems if we cannot verify when the assumptions are satisfied. It would therefore be much more reassuring if we could find some criteria that implies lower semi-continuity for the functional. This is what we will do next. As so often in mathematics we will try to understand a complicated situation by constructing an example easy enough for us to explicitly calculate it. In PDE theory that usually means construction a one dimensional example since the power of one dimensional calculus allows us to do most calculations explicitly in one dimension.

Example: Consider the one-dimensional minimizing problem

$$\text{minimize } J_F(f(x)) = \int_0^1 F(f'(x)) dx \tag{5}$$

[4]See the appendix to this chapter for a brief explanation of this property.

in the set

$$K = \left\{ f \in W^{1,2}(0,1);\ f(0) = 0 \text{ and } f(1) = 1 \right\}.$$

We need to choose our function $F(\cdot)$ which we choose quite randomly to be the function with the graph

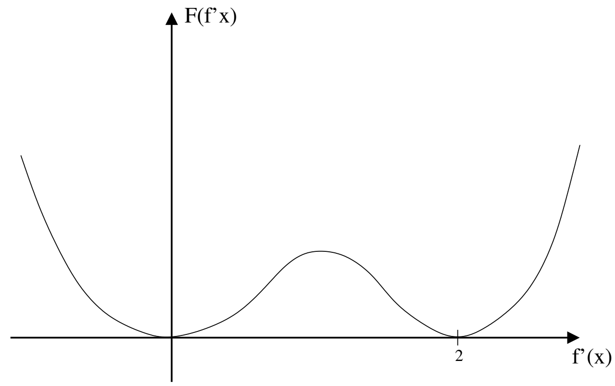


Figure 2: *The graph of the function $F(\cdot)$.*

Since $F(\cdot) \geq 0$ we can conclude that $J(f) \geq 0$ for all functions $f \in K$. But if

$$f'(x) = \begin{cases} 0 \text{ if } x \in A \\ 2 \text{ if } x \notin A \end{cases} \tag{6}$$

for some set A then the energy $J_F(f'(x)) = 0$ since $F(0) = F(2) = 0$. Thus any function $f(x)$ of the form (6) will be a minimizer to (5). Notice that such a minimizer can arbitrarily well approximate (in $C^0([0,1])-$norm) any function $g(x)$ satisfying $0 \leq g' \leq 2$. This can be clearly seen in the following picture:

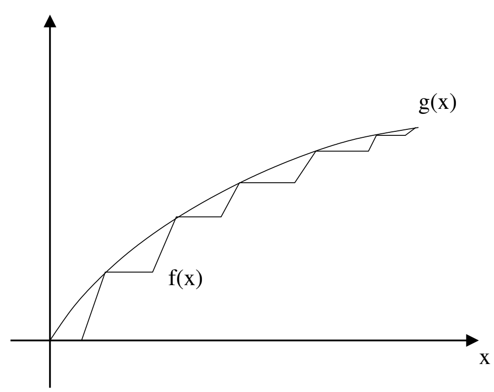


Figure 3: *Graphic representation of how a function $f(x)$ whose derivative takes the values* 0 *and* 2 *approximates an arbitrary function $g(x)$ with derivative* $0 \leq g'(x) \leq 2$.

This implies that for any function $g(x) \in K$ such that $0 \leq g'(x) \leq 2$ we can find a sequence $f^j \in K$ such that $f^j \to g$ uniformly and $J(f^j(x)) = 0$. But

$J(g(x))$ may very well be strictly positive, for instance if $g(x) = x$. Thus the functional $J_F(f)$ defined in (5) is not lower semi-continuous.[5]

The question we need to ask is: *Is the problem that the function F is zero at two different points?* A simple example shows that that is not the case.

Consider for instance the one dimensional minimization problem

$$\text{minimize } J_G(f(x)) = \int_0^1 G(f'(x))dx$$

in the set

$$K = \left\{ f \in W^{1,p}(0,1);\ f(0) = 0 \text{ and } f(1) = 1 \right\},$$

where the function G is given by the graph:

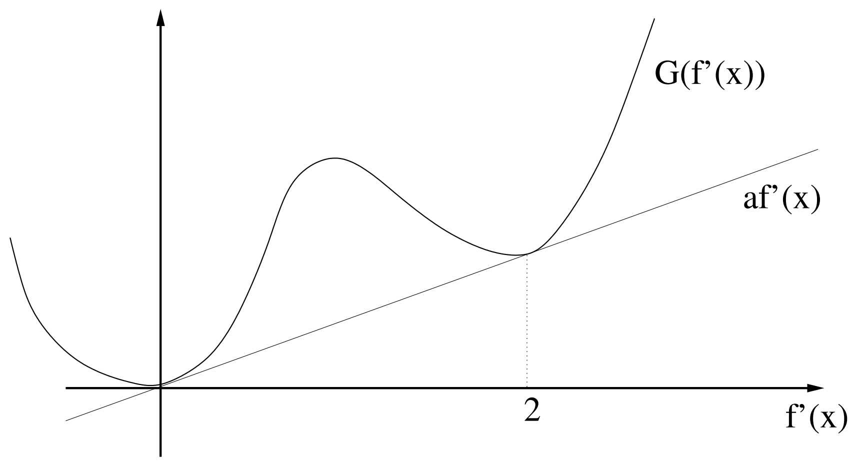


Figure 4: *The graph of the function $G(f'(x))$ and $af'(x)$.*

If we subtract the linear function $af'(x)$ from $G(f'(x))$ we will get a function with graph looking like the one in Figure 3; we may even assume that $G(f'(x)) = F(f'(x)) + af'(x)$. This leads to

$$J_G(f'(x)) = \int_0^1 G(f'(x))dx = \int_0^1 F(f'(x))dx + a\int_0^1 f'(x)dx =$$

$$= J_F(f'(x)) + af(1) - af(0),$$

where we used an integration by parts in the last equality. Since $af(1) - af(0) = a$ for all $f \in K$ we can conclude that

$$J_G(f'(x)) = J_F(f'(x)) + a \text{ for all } f \in K.$$

And since J_F and J_G only differ by a constant we can conclude that J_G cannot have a minimizer since J_F does not have a minimizer. □

[5]Notice that the lower semi-continuity of J_F is not really related to the continuity of F. We may very well, as in the example, have that F is a continuous function but J_F is not continuous on the space $W^{1,2}(D)$.

What conclusion can we draw from this example? The reason that there a minimizer to $\int_0^1 G(f'(x))dx$ does not exist was that we could touch the graph of G from below, at two different points, by a linear function. That is: G is not convex. Clearly convexity is a necessary condition for a minimizer to exist, at least for minimization in $\mathbb{R}$. It turns out that convexity is the assumption needed in any dimension $\mathbb{R}^n$, not just for examples in $\mathbb{R}$, for the functional $J(u)$ to be lower semi-continuous.[6] We are ready to formulate and prove an existence theorem for minimizers.

Theorem 2.2. *Assume that D is a given bounded domain*[7] *and that $F : \mathbb{R}^n \mapsto \mathbb{R}$ is a continuously differentiable function satisfying:*

1. *There exists a constant C such that $F(\nabla u) \geq -C$*
2. *$\int_D F(\nabla u) \to \infty$ as $\|\nabla u\|_{L^2(D)} \to \infty$*
3. *$F(p)$ is convex:*

$$F(\boldsymbol{q}) \geq F(\boldsymbol{p}) + F'(\boldsymbol{p})(\boldsymbol{q} - \boldsymbol{p}) \qquad \text{for any } \boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}.$$

Then for any closed (under weak limits) sub-set $K \subset W^{1,2}(D)$ there exists a function $u \in K$ such that

$$\int_D F(\nabla u)dx = \inf_{v \in K} \int_D F(\nabla v)dx.$$

Proof: The proof follows the same steps as the proof in the one dimensional case.

Since $F(\nabla u) \geq -C$ we can conclude that for any $u \in K$

$$J(u) = \int_D F(\nabla u(x))dx \geq -C \int_D dx = -C|D|,$$

where $|D|$ denotes the volume of the set $|D|$. Thus the set of values $J(u)$ can obtain is bounded from below and therefore the the number $m = \inf_{u \in K} J(u)$ exists and is well defined.

We may therefore find a sequence u^j such that $J(u^j) \to m$. Notice that since $J(u) \to \infty$ as $\|\nabla u\|_{L^2(D)} \to \infty$ it follows that $\|\nabla u^j\|_{L^2(D)}$ is bounded. By weak compactness there is a sub-sequence u^{j_k} and a function $u^0 \in W^{1,2}(D)$ such that $u^{j_k} \rightharpoonup u^0$ in $W^{1,2}(D)$. Since K is closed it follows that $u^0 \in K$. There is no loss of generality to assume that the sub-sequence u^{j_k} is the full sequence u^j.

We need to verify that $J(u^0) = m$. To that end we calculate as $j \to \infty$

$$m \leftarrow \int_D F(\nabla u^j)dx = \int_D F(\nabla u^0 + \nabla(u^j - u^0))dx \geq \{\text{covexity}\} \geq$$

[6]Here we are talking about minimization for scalar valued functions $u(x)$. If $u(x)$ is vector valued there exists many different notions of convexity (quasi-convexity, poly-convexity et.c.) that implies existence in different situations. We will not consider minimization problems with vector valued functions in this course.

[7]By a domain we mean an open set in $\mathbb{R}^n$.

$$\geq \int_D F(\nabla u^0) + \int_D \underbrace{F'(\nabla u^0)\cdot(\nabla u^j}_{\to F'(\nabla u^0)\cdot\nabla u^0} - \nabla u^0)dx \to \tag{7}$$

$$\to \int_D F(\nabla u^0) + \int_D F'(\nabla u^0)\cdot(\nabla u^0 - \nabla u^0)dx = \int_D F(\nabla u^0)dx.$$

The calculation (7) proves that $J(u^0) = m = \inf_{u\in K} J(u)$. $\square$

Proposition 2.1. *If the function $F(p)$ in Theorem 2.2 is strictly convex*

$$F(\boldsymbol{q}) > F(\boldsymbol{p}) + F'(\boldsymbol{p})(\boldsymbol{q}-\boldsymbol{p}) \quad \textit{for any } \boldsymbol{p},\boldsymbol{q}\in\mathbb{R},$$

for all $\boldsymbol{p}\neq\boldsymbol{q}$ and the domain D is connected. then the minimizer is unique up to additive constants. In particular the minimizer is unique among all functions with the same boundary data.

Proof: Assume that we have two minimizers $u(x)\in K$ and $v(x)\in K$ then by strict convexity and that both minimizers have the same energy

$$0 = \int_D F(\nabla u(x))dx - \int_D F(\nabla v(x))dx \geq \int_D F'(\nabla u)\cdot(\nabla(v-u))dx, \tag{8}$$

with equality only if $\nabla u(x) = \nabla v(x)$.

Since u is a minimizer and K is convex, which implies that $u(x)+t(v(x)-u(x))\in K$ for $t\in[0,1]$,

$$0 \leq \int_D F(\nabla u(x) + t\nabla(v(x)-u(x)))dx. \tag{9}$$

Dividing by $t>0$ and letting $t\to 0$ in (9) gives

$$0 \leq \lim_{t\to 0^+}\frac{1}{t}\int_D F(\nabla u(x)+t\nabla(v(x)-u(x)))dx = \int_D F'(\nabla u)\cdot(\nabla(v-u))dx.$$

Comparing this to (8) we see that we must have equality in (8). But as we remarked earlier we only have equality in (8) if $\nabla u(x) = \nabla v(x)$. It follows that $u(x)-v(x)$ is constant.

If $u(x)=v(x)$ on ∂D then clearly $u(x)-v(x)=0$ and it follows that the minimizer is unique among the functions with the same boundary data. $\square$

The importance of the existence theorem and the uniqueness proposition is, of course, the following Theorem from Evans' book.

Theorem 2.3. *[Dirichlet's Principle.] Assume that $u\in C^2(\overline{D})$ solves*

$$\begin{array}{ll} \Delta u(x) = f(x) & \textit{in } D \\ u(x) = g(x) & \textit{on } \partial D. \end{array} \tag{10}$$

Then

$$\int_D \left(\frac{1}{2}|\nabla u(x)|^2 - u(x)f(x)\right)dx = \inf_{w\in K}\int_D\left(\frac{1}{2}|\nabla w(x)|^2 - w(x)f(x)\right)dx, \tag{11}$$

where $K=\{w\in C^2(\overline{D});\ w(x)=g(x) \text{ on } \partial D\}$.

Conversely, if $u\in K$ satisfies (11) then $u(x)$ solves (10).

The minimizer $u(x)$ of the Dirichlet energy therefore is a very good candidate to be the solution to Laplace equation - and is indeed the unique solution if it is C^2. It is indeed the case that the minimizer of the Dirichlet energy is C^2 and we thus have a new way to find solutions to Laplace equation. Furthermore, the existence proofs you have seen previously in this course (by means of explicitly constructing a Green's function) only works for very nice domains, such as balls and half-spaces, where the explicit calculations can be carried out. The variational existence theorem works for any domain. However, one has to impose some restrictions on the domain in order to assure that the minimizer assumes the right boundary data, see Corollary 2.2.

Exercises.

1. [The Maximum principle.] You already know that if $u(x)$ is harmonic in D then $\sup_{x\in D} u(x) = \sup_{x\in\partial D} u(x)$. Prove this using that harmonic functions minimize the Dirichlet energy $\int_D |\nabla u(x)|^2 dx$ among all functions with the same boundary values.

   Hint: *Let* $M = \sup_{x\in\partial D} u(x)$ *and consider the energy of the function* $u_M(x) = \min(u(x), M,)$. *Show that* u_M *has less than or equal Dirichlet energy as* u *and use uniqueness of minimizers.*

2. * [Comparison principle.] Assume that $u(x)$ and $v(x)$ are harmonic in D and that $u(x) \le v(x)$ on ∂D. Use the variational formulation of the Dirichlet problem to prove that $u(x) \le v(x)$ in D.

   Hint: *See previous exercise.*

3. [Generalizations.] Theorem 2.3 states that finding minimizers to the Dirichlet energy is the same as solving $\Delta u(x) = 0$. However, the real strength of the calculus of variations is that it easily generalizes to a wide variety of problems.

   (a) Assume that $a(x) > 0$ and $a(x) \in C^1(D)$. Show that there exists a minimizer to $\int_D a(x)|\nabla u(x)|^2 dx$ in the set $K = \{u \in W^{1,2}(D);\ u(x) = f(x) \text{ on } \partial D\}$. What partial differential equation does the minimizer solve?

   Hint: *Follow the proof of Theorem 2.3.*

   (b) Assume that we have a minimizer[8] to $\int_D |\nabla u(x)|^p$ in $K = \{u \in W^{1,p}(D);\ u(x) = f(x) \text{ on } \partial D\}$ for some $1 < p < \infty$. Assume furthermore that the minimizer $u(x) \in C^2(D)$. What partial differential equation does $u(x)$ solve?[9]

[8] This is actually just as easy to prove as the theorem above. At least if one knows a little basic functional analysis that I do not feel that we have time for right now.

[9] The PDE is called the $p-$laplacian and the function $u(x)$ is called $p-$harmonic. This example is interesting since the $p-$laplacian is non-linear and one can not construct solutions using the Green-function methods used in Evans to find solutions.

(c) Finally, consider a minimizer $u(x)$ to $\int_D |\Delta u(x)|^2 dx$. What PDE does $u(x)$ solve?[10]

4. * [NEUMANN DATA.] Let $u(x)$ minimize the Dirichlet energy $\int_{B_1(0)} |\nabla u(x)|^2 dx$ in the set

$$K = \{u \in W^{1,2}(B_1(0));\ u(x) = f(x) \text{ on } \partial B_1(0) \cap \{x_n \leq 0\}\}.$$

Notice that the boundary data is only imposed on the negative half of the ball. Show that

$$\frac{\partial u(x)}{\partial \nu} = 0 \text{ on } \partial B_1(0) \cap \{x_n > 0\},$$

where $\nu = \frac{x}{|x|}$ is the outer normal of $\partial B_1(0)$.

HINT: *Make variations $w(x) = u(x) + t\phi(x)$ in the proof of Theorem 2.3 where $\phi(x)$ is not necessarily zero on $\partial B_1(0) \cap \{x_n > 0\}$.*

5. ** Find an $F \in C^1(\mathbb{R}^n \mapsto \mathbb{R})$ so that the functional $J(u) = \int_D F(\nabla u(x))dx$ admits several minimizers.

HINT: *What is the difference in the assumptions in the existence Theorem 2.2 and the uniqueness Proposition 2.1?*

2.1 Appendix. A painfully brief introduction to Sobolev spaces and weak convergence.

In this appendix we gather some facts about the convergence of functions that we need in order to show the existence of minimizers. Ideally we would prove all the results in the appendix - but we will not strive for the ideal. Some of the results belong properly to functional analysis and measure theory[11] and would take us to far off topic. We will however try to motivate some of the results informally.

The central concept in analysis is convergence. We see this already in a first calculus course; when we learn about the convergence of difference quotients in order to define derivatives and the convergence of Riemann sums in order to define the integral. The next order of sophistication is to consider the convergence of functions.

What does it mean for a sequence of functions f_j to converge to a function f_0? The answer to that question is manifold and it depends much on the situation

[10] This is an important example in $\mathbb{R}^2$ since it is a model for the bending of a thin metal plate.

[11] If you take the course "Advanced real analysis I" you will see proper proofs of some of the theorems in this appendix.

which kind of convergence is relevant. In the calculus of variations we are often interested in sequences of functions $u^j(x)$ such that

$$\int_D |\nabla u^j(x)|^2 dx \leq C$$

for some given domain D and constant C. That is functions whose derivative is square integrable. The natural space to consider would therefore be

Definition 2.1. *Let $D \subset \mathbb{R}^n$ be a given set. Then we write $L^2(D)$ for the set of all functions $u(x)$ defined on D such that*

$$\|u\|_{L^2(D)} = \left(\int_D |u(x)|^2 dx\right)^{\frac{1}{2}} < \infty.$$

Remark: *The L in $L^2(D)$ stands for Lebesgue since for these spaces are always considering the Lebesgue integral. However, if you are not familiar with the Lebesgue integral you can think of the integral as being a Riemann integral. There are some instances where we will use a fact that is true only for the Lebesgue integral but not for the Riemann integral - you will have to accept those facts.*

The importance of the space $L^2(D)$ comes form the following Theorem.

Theorem 2.4. *The Space $L^2(D)$ with the norm $\|u\|_{L^2(D)}$ is a complete space. That is if $u^j \in L^2(D)$ is a Cauchy sequence, $\lim_{j,k\to\infty} \|u^j - u^k\|_{L^2(D)} = 0$, then there exists a function $u^0 \in L^2(D)$ such that $\lim_{j\to\infty} \|u^j - u^0\|_{L^2(D)} = 0$.*

Remark: *The completeness is not true for Riemann integrable functions. For instance the sequence $u^j(x) \in L^(0,1)$ defined so that $u^j(x) = 1$ if x is one of the first j rational numbers (in some ordering) and $u^j(x) = 0$ else. Then the Riemann integral $\int_0^1 |u^j|^2 dx = 0$ and u^j converges point-wise to a function $u^0(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ that is not Riemann integrable.*

In $L^2(D)$ we say that $u^j(x) \to u^0(x)$ if $\|u^j - u^0\|_{L^2(D)} \to 0$ as $j \to \infty$. We would want this convergence to have some good properties. We would in particular like the Bolzano-Weierstrass Theorem[12] to hold. Let us briefly remind ourselves of the Bolzano-Weierstrass Theorem in $\mathbb{R}^n$.[13]

Theorem 2.5. *Let $K \subset \mathbb{R}^n$ be a closed and bounded set and $\{x^j\}_{j=1}^\infty$ be a sequence such that $x^j \in K$. Then there exists a sub-sequence $\{x^{j_k}\}_{k=1}^\infty$ of $\{x^j\}_{j=1}^\infty$ such that $\lim_{k\to\infty} x^{j_k}$ exists.*

[12] That is every sequence u^j such that $\|u^j\|_{L^2(D)} \leq C$ for some constant C has a convergent sub-sequence.

[13] I am a strong believer that there is no difference between first year calculus courses and PhD level courses and must therefore consistently refer back to undergraduate stuff in all my courses. But you shouldn't complain - I refer to PhD level stuff in my first year undergraduate courses as well so you are better off than my first year students...

Sketch of the Proof: Let $x^j = (x_1^j, x_2^j, ..., x_n^j)$ then by the Bolzano-Weierstrass Theorem in $\mathbb{R}$ we may find a sub-sequence, which we may denote $\{x_1^{j_{k,1}}\}_{k=1}^\infty$, of $\{x_1^j\}_{j=1}^\infty$ such that $\{x_1^{j_{k,1}}\}_{k=1}^\infty$ converges.

Again by the one dimensional Bolzano-Weierstrass Theorem we may find a subsequence of $j_{k,1}$, lets denote it $j_{k,2}$, such that $\{x_2^{j_{k,2}}\}_{k=1}^\infty$ converges. We may then find a subsequence of $j_{k,2}$, lets denote it $j_{k,3}$, such that $\{x_3^{j_{k,3}}\}_{k=1}^\infty$ converges et.c.

In the end we find a sequence $j_k = j_{k,n}$ such that $\{x_n^{j_k}\}_{k=1}^\infty$ converges. But since j_k, by construction, is a sub-sequence of each sequence $\{j_{k,l}\}_{k=1}^\infty$, $l = 1, 2, ..., n$ it follows that

$$\lim_{k\to\infty} x_l^{j_k} = x_l^0 \quad \text{for} \quad l = 1, 2, ..., n.$$

This finishes the proof. □

In order to gain some feeling for the convergence properties of functions in $L^2(D)$ we need to make some calculations. Before we start our investigation into convergence in L^2 we remind ourselves of Paresval's Theorem and the Cauchy-Schwartz inequality.

Theorem 2.6. *Let $u(x), v(x) \in L^2(-\pi, \pi)$ and*

$$u(x) = \frac{a_0(u)}{\sqrt{2\pi}} + \sum_{k=1}^\infty a_k(u)\frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^\infty b_k(u)\frac{\sin(kx)}{\sqrt{\pi}}$$

and

$$v(x) = \frac{a_0(v)}{\sqrt{2\pi}} + \sum_{k=1}^\infty a_k(v)\frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^\infty b_k(v)\frac{\sin(kx)}{\sqrt{\pi}}$$

then

- [Parseval's Theorem.]

$$\int_{-\pi}^{\pi} |u(x)|^2 dx = a_0(u)^2 + \sum_{k=1}^\infty \left(a_k(u)^2 + b_k(u)^2\right).$$

- [Cauchy-Schwartz Inequality.] *For any two functions $g, h \in L^2(D)$ the following inequality holds*

$$\int_D g(x)h(x)dx \le \left(\int_D |g(x)|^2 dx\right)^{1/2} \left(\int_D |h(x)|^2 dx\right)^{1/2} \tag{12}$$

this can be formulates for $u(x)$ and $v(x)$ as

$$\int_{-\pi}^{\pi} u(x)v(x)dx = a_0(u)a_0(v) + \sum_{k=1}^\infty (a_k(u)a_k(v) + b_k(u)b_k(v)).$$

Parseval's Theorem allows us to view $L^2(-\pi,\pi)$ as an infinite dimensional vector space with basis $\frac{\sin(kx)}{\sqrt{\pi}}$ and $\frac{\cos(kx)}{\sqrt{\pi}}$. This allows us to make some of the calculations more explicit in the one dimensional setting.[14] Next we provide an example that shows that the Bolzano-Weierstrass Theorem does not hold in $L^2(-\pi,\pi)$.

Example: *The Bolzano-Weierstrass Theorem does not hold on $L^2([-\pi,\pi])$. In particular, we may find a sequence of functions $u^j \in L^2([-\pi,\pi])$ such that $\|u^j\|_{L^2([-\pi,\pi])} = 1$ but u^j does not contain any convergent sub-sequence.*

Proof of the example: We will use some Fourier analysis. For a function $u(x)$ we will write

$$u(x) = \frac{a_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} a_k \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} b_k \frac{\sin(kx)}{\sqrt{\pi}},$$

where

$$a_k = \int_{-\pi}^{\pi} u(x)\frac{\cos(kx)}{\sqrt{\pi}}dx \quad \text{and} \quad b_k = \int_{-\pi}^{\pi} u(x)\frac{\sin(kx)}{\sqrt{\pi}}dx.$$

The sequence $u^j(x) = \frac{1}{\sqrt{\pi}}\cos(jx)$ will satisfy $\|u^j\|_{L^2(-\pi,\pi)} = 1$. The only non-zero Fourier-coefficient of u^j is $a_j = 1$. We claim that $u^j(x)$ cannot have any convergent subsequence. To see this we assume the contrary; that $u^j \to u^0$, for a sub-sequence, where

$$u^0(x) = \frac{a_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} a_k \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} b_k \frac{\sin(kx)}{\sqrt{\pi}}.$$

By Parseval's Theorem

$$\int_{-\pi}^{\pi} |u^j(x) - u^0(x)|^2 dx = \left(a_0^2 + \sum_{k=1, k\neq j}^{\infty} \left(a_k^2 + b_k^2\right) + (1-a_j)^2 + b_j^2 \right)$$

$$\geq \left(a_0^2 + \sum_{k=1}^{j-1} \left(a_k^2 + b_k^2\right) \right).$$

Since, if $u^j \to u^0$, then the right hand side tends to zero (for some sub-sequence) we can conclude that if $u^k \to u^0$ then $a_k = 0$ and $b_k = 0$ for all k. That is $u^0 = 0$. But this implies that

$$\|u^j - u^0\|_{L^2(-\pi,\pi)} = \|u^j\|_{L^2(-\pi,\pi)} = 1$$

[14] Any basic course in functional analysis will teach you that $L^2(D)$ is a Hilbert space and thus has a basis. The difference with $L^2(-\pi,\pi)$ is that we may write down the basis explicitly with familiar trigonometric functions.

which contradicts $\|u^j - u^0\|_{L^2(-\pi,\pi)} \to 0$. We can conclude that $u^j \not\to u^0$ for any subsequence. □

The above example shows that we cannot hope for a Bolzano-Weierstrass Theorem for $L^2(D)$ for any domain D.

However, we are still able to salvage something from the proof of the finite dimensional Bolzano-Weierstrass Theorem to the infinite dimensional case. If we assume that $u^j(x)$ is a sequence of functions in $L^2(-\pi,\pi)$ with the Fourier expansion

$$u^j(x) = \frac{a_0^j}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} a_k^j \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} b_k^j \frac{\sin(kx)}{\sqrt{\pi}}$$

and $\|u^j\|_{L^2(-\pi,\pi)} \leq C$ then the sequence of numbers $\{c_k^j\}_{j=1}^{\infty}$ ($c_k^1 = a_k^1, c_k^2 = b_k^1, c_k^3 = a_k^2, c_k^4 = b_k^2, c_k^5 = a_k^2...$) must be bounded: $|c_k^j| \leq C$ and $|c_k^j| \leq C$. This means that we can find a sub-sequence $c_1^{l_{1,j}}$ of c_1^j that converges $c_1^{l_{1,j}} \to c_1^0$, a subsequence $l_{2,j}$ of $l_{1,j}$ such that $c_2^{l_{2,j}} \to a_2^0$ and inductively a subsequence $l_{k,j}$ of $l_{k-1,j}$ such that $c_k^{l_{k,j}} \to c_k^0$.

By choosing the diagonal sequence $l_j = l_{j,j}$, just as in the Arzela-Ascoli Theorem, we see that $a_k^{l_j} \to a_k^0$ and $b_k^{l_j} \to b_k^0$ for any k. Thus there exists a subsequence u^{l_j} whose Fourier coefficients all converge to a_k^0 and b_k^0 respectively. We may thus find a subsequence of any bounded sequence $u^j \in L^2(-\pi,\pi)$ such that all the Fourier coefficients converge - it is not what we want but it will have to do.

The mode of convergence of the sequence above is called weak convergence. Since weak convergence is a much more general concept than something that applies only for $L^2(-\pi,\pi)$ we will give the classical definition of weak convergence. Later we will prove that the above *convergence of all the Fourier coefficients* is indeed weak convergence defined as follows.

Definition 2.2. *We say that u^j converges weakly in $L^2(D)$ to u^0, or simply write $u^j \rightharpoonup u^0$, if for every function $v \in L^2(D)$*

$$\int_D u^j(x)v(x)dx \to \int_D u^0(x)v(x)dx. \tag{13}$$

Remark: *Observe that we use the symbol $\rightharpoonup$ and not $\to$ for weak convergence. It is also important that we have the same function $v(x)$ in the integrals (13).*

Our first Lemma for weakly converging functions is.

Lemma 2.1. *Let $u^j \rightharpoonup u^0$ in $L^2(D)$ for some domain D. Then*

$$\liminf_{j\to\infty} \|u^j\|_{L^2(D)} \geq \|u^0\|_{L^2(D)}.$$

Proof: Consider

$$\liminf_{j\to\infty} \left(\|u^j\|_{L^2(D)} - \|u^0\|_{L^2(D)}^2\right) = \liminf_{j\to\infty} \int_D \left(|u^j(x)|^2 - |u^0(x)|^2\right) dx =$$

$$= \liminf_{j\to\infty} \int_D \left(|u^j(x)|^2 - |u^0(x)|^2 - 2u^j(x)u^0(x) + 2|u^0(x)|^2\right) dx = \tag{14}$$

$$= \liminf_{j\to\infty} \int_D \left(|u^j(x) - u^0(x)|^2\right) dx \geq 0,$$

where we used that, by weak convergence,

$$\lim_{j\to\infty} \int_D \left(u^j(x)u^0(x) - |u^0(x)|^2\right) dx = \int_D \left(u^0(x)u^0(x) - |u^0(x)|^2\right) dx = 0,$$

in the equality leading to (14). The Lemma follows. □

So far we have not shown that *any sequence whatsoever* of functions $u^j \in L^2(D)$ converges weakly - but we suspect that the convergence of the Fourier coefficients should imply weak convergence. But as a matter of fact we regain the Bolzano-Weierstrass Theorem if we consider weak convergence instead of strong.

Theorem 2.7. [Weak Compactness Theorem.] *Let $u^j(x) \in L^2(D)$, $D \subset \mathbb{R}^n$, be a bounded sequence*[15]*. Then there exists a sub-sequence, which we still denote u^j, that converges weakly $u^j \rightharpoonup u^0$ for some function $u^0 \in L^2(D)$.*

Sketch of proof: The Theorem is formulated for any domain $D \subset \mathbb{R}^n$. We will for the sake of simplicity sketch the proof in $L^2(-\pi,\pi)$. The general proof is based on the same ideas - but would use more functional analysis that we neither want to prove not presuppose for this course.

Step 1: *The set up.*

We assume that u^j is a sequence of functions with the Fourier expansions

$$u^j(x) = \frac{a_0^j}{2} + \sum_{k=1}^{\infty} a_k^j \cos(kx) + \sum_{k=1}^{\infty} b_k^j \sin(kx).$$

Since, by assumption, $\|u^j\|_{L^2(-\pi,\pi)}$ is bounded it follows that $|a_k^j|$ and $|b_k^j|$ are bounded. By the one dimensional Bolzano-Weierstrass Theorem we can find a subsequence, $u^{j_{0,k}}$ of u^j such that $a_0^{j_{0,k}} \to a_0^0$ and $b_0^{j_{0,k}} \to b_0^0$. Choosing a subsequence again, which we denote $u^{j_{1,k}}$, we can assure that $a_1^{j_{1,k}} \to a_1^0$ and $b_1^{j_{1,k}} \to b_1^0$. Continuing inductively we may define the diagonal sequence $j_{k,k}$ for which $a_l^{j_{k,k}} \to a_l^0$ and $b_l^{j_{k,k}} \to b_l^0$ for every l.

We will denote

$$u^0(x) = \frac{a_0^0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} a_k^0 \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} b_k^0 \frac{\sin(kx)}{\sqrt{\pi}}.$$

We claim that $u^{j_{k,k}} \rightharpoonup u^0$. In order to simplify notation we will write $j = j_{k,k}$ knowing that we have choses a sub-sequence already.

[15]When we talk about bounded sequences in L^2 we always mean sequences with bounded norm: $\|u^j\|_{L^2(-\pi,\pi)} \leq C_u$ for some constant C_u independent of j.

We need to show (13) for any function $v \in L^2(-\pi,\pi)$. We define the Fourier-coefficients of v according to

$$v(x) = \frac{a_0^v}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} a_k^v \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} b_k^v \frac{\sin(kx)}{\sqrt{\pi}}.$$

Step 2: *Proof if $a_k^v = 0$ and $b_k^v = 0$ if $k > M$.*

Proof of Step 2: By Pareval's Theorem we get

$$\int_{-\pi}^{\pi} \left(u^j(x) - u^0(x)\right) v(x)dx = \sum_{k=0}^{\infty} (a_k^j - a_k^0) a_k^v + \sum_{k=1}^{\infty} (b_k^j - b_k^0) b_k^v = \tag{15}$$

$$= \left\{ \begin{array}{l} \text{use } a_k^v, b_k^v = 0 \\ \text{for } k > M \end{array} \right\} = \sum_{k=0}^{M} (a_k^j - a_k^0) a_k^v + \sum_{k=1}^{M} (b_k^j - b_k^0) b_k^v \to 0$$

where we used that $a_k^j - a_k^0 \to 0$ and $b_k^j - b_k^0 \to 0$. This implies (13).

Step 3: *Proof for general $v \in L^2(-\pi,\pi)$.*

Proof of Step 3: We may write $v(x) = v_M(x) + w_M(x)$ where

$$v_M(x) = \frac{a_0^v}{\sqrt{2\pi}} + \sum_{k=1}^{M} a_k^v \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{M} b_k^v \frac{\sin(kx)}{\sqrt{\pi}}$$

and

$$w_M = \sum_{k=M+1}^{\infty} a_k^v \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=M+1}^{\infty} b_k^v \frac{\sin(kx)}{\sqrt{\pi}}.$$

Since $v \in L^2(-\pi,\pi)$ we know, from Parseval's Theorem, that the series

$$\sum_{k=1}^{\infty} (a_k^v)^2 + \sum_{k=1}^{\infty} (b_k^v)^2$$

converges. We may therefore, for any $\epsilon > 0$, choose M so that

$$\|w_M\|^2_{L^2(-\pi,\pi)} = \sum_{k=M+1}^{\infty} (a_k^v)^2 + \sum_{k=M+1}^{\infty} (b_k^v)^2 < \epsilon^2. \tag{16}$$

It follows that

$$\left| \int_{-\pi}^{\pi} \left(u^j(x) - u^0(x)\right) v(x)dx \right| =$$

$$= \left| \int_{-\pi}^{\pi} \left(u^j(x) - u^0(x)\right) v_M(x)dx \right| + \left| \int_{-\pi}^{\pi} \left(u^j(x) - u^0(x)\right) w_M(x)dx \right| \leq \tag{17}$$

$$\leq \left| \underbrace{\int_{-\pi}^{\pi} \left(u^j(x) - u^0(x)\right) v_M(x)dx}_{\to 0 \text{ by Step 2}} \right| + \|u^j - u^0\|_{L^2(-\pi,\pi)} \|w_M\|_{L^2(-\pi,\pi)},$$

where we used Cauchy-Schwartz inequality in the last step of the calculation.

Notice that by the triangle inequality and Lemma 2.1

$$\|u^j - u^0\|_{L^2(-\pi,\pi)} \leq \|u^j\|_{L^2(-\pi,\pi)} + \|u^0\|_{L^2(-\pi,\pi)} \leq 2C_u \tag{18}$$

where C_u is the bound on $\|u^j\|_{L^2(-\pi,\pi)}$.

By choosing M large enough we can, by (16) and (18), make the term

$$\|u^j - u^0\|_{L^2(-\pi,\pi)}\|w_M\|_{L^2(-\pi,\pi)} < \epsilon. \tag{19}$$

Passing to the limit in (17) and using (19) we can conclude that

$$\lim_{j\to\infty}\left|\int_{-\pi}^{\pi}\left(u^j(x) - u^0(x)\right)v(x)dx\right| < \epsilon,$$

for any $\epsilon > 0$. The Theorem follows. $\square$

When we do calculus of variations we are really interested in functions whose derivatives are in the space $L^2(D)$. We need the following definitions.

Definition 2.3. *Let u be integrable in some domain $D \subset \mathbb{R}^n$. Then if there exists an integrable function $w(x)$ such that*

$$\int_D \frac{\partial v(x)}{\partial x_i}u(x)dx = -\int_D v(x)w(x)dx \text{ for every } v(x) \in C_c^1(D) \tag{20}$$

then we say that $u(x)$ is weakly differentiable in x_i and that $w(x)$ is the weak x_i-derivative of $u(x)$ and write $\frac{\partial u(x)}{\partial x_i} = w(x)$.

Remark: *Notice that the definition is made so as the partial integration formula works. In particular, if $u(x)$ is weakly differentiable in x_i then (20) become the normal integration by parts formula*

$$\int_D \frac{\partial v(x)}{\partial x_i}u(x)dx = -\int_D v(x)\frac{\partial u(x)}{\partial x_i}dx.$$

It follows directly that every continuously differentiable function is weakly differentiable.

Definition 2.4. *Let $u(x) \in L^2(D)$ be weakly differentiable in every direction x_i, $i = 1, 2, ..., n$ and the weak derivatives $\frac{\partial u}{\partial x_i} \in L^2(D)$ for all $i = 1, 2, ..., n$. Then we say that $u \in W^{1,2}(D)$. We call the space of all such functions equipped with the norm*

$$\|u\|_{W^{1,2}(D)} = \left(\int_D |u(x)|^2dx + \int_D |\nabla u(x)|^2dx\right)^{1/2}, \tag{21}$$

where $\nabla u(x) = \left(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, ..., \frac{\partial u(x)}{\partial x_n}\right)$.

We will also write $W^{k,2}(D)$ for all functions defined on D such that all weak derivatives up to order k exists and

$$\|u\|_{W^{k,2}(D)} = \left(\sum_{|\alpha|\leq k} \int_D |D^\alpha u|^2 \right)^{1/2} < \infty,$$

where the summation is over all mutiindexes α of length $|\alpha| \leq k$.

Remark: *Often in analysis one uses the Sobolev space $W^{k,p}(D)$ with norm:*

$$\|u\|_{W^{k,p}(D)} = \left(\sum_{|\alpha|\leq k} \int_D |D^\alpha u|^p \right)^{1/p} < \infty.$$

We will not use this space in this course.

Lemma 2.2. *The space $W^{1,2}(D)$ with norm (21) is a complete space.*

Furthermore every bounded sequence of functions $u^j \in W^{1,2}(D)$ has a subsequence u^{j_k} so that $u^{j_k} \rightharpoonup u^0$ and $\frac{\partial u^{j_k}}{\partial x_i} \rightharpoonup \frac{\partial u^0}{\partial x_i}$ in $L^2(D)$ for some $u^0 \in W^{1,2}(D)$.

Remark: *We say that the subsequence u^{j_k} converges weakly to u^0 in $W^{1,2}(D)$, written $u^{j_k} \rightharpoonup u^0$ in $W^{1,2(D)}$.*

Proof: That $W^{1,2}(D)$ is complete follows from the same statement for $L^2(D)$, Theorem 2.4.

By Theorem 2.7 we can clearly extract a subsequence such that u^{j_k} and $\frac{\partial u^{j_k}}{\partial x_i}$, for all $i = 1, 2, ..., n$, converges weakly. We need to show that the limit of the partial derivatives converges to the partial derivatives of the limit $u^{j_k} \rightharpoonup u^0$. That is easy. For any $\phi \in C^1(D)$ we have

$$\int_D \frac{\partial \phi(x)}{\partial x_i} u^0(x)dx \leftarrow \int_D \frac{\partial \phi(x)}{\partial x_i} u^{j_k}(x)dx = \tag{22}$$

$$-\int_D \phi(x) \frac{\partial u^{j_k}(x)}{\partial x_i} dx \to -\int_D \phi(x) \lim_{j_k\to\infty} \frac{\partial u^{j_k}(x)}{\partial x_i} dx,$$

Since (22) holds for every ϕ it follows that $\frac{\partial u^{j_k}(x)}{\partial x_i} \rightharpoonup \frac{\partial u^0(x)}{\partial x_i}$. This finishes the proof. $\square$

We need one final, and rather subtle, concept regarding Sobolev spaces in order to use them in the calculus of variations.

Theorem 2.8. [TRACES.] *Let D be a bounded domain with continuously differentiable boundary. Then there exists an operator*

$$T : W^{1,2}(D) \mapsto L^2(\partial D)$$

that assigns boundary values (in the trace sense) of $u \in W^{1,2}(D)$ onto the boundary ∂D.

Furthermore $Tu = u\lfloor_{\partial D}$ for all functions $u \in C(\overline{D}) \cap W^{1,2}(D)$.

Before we prove this theorem we will try to motivate its importance. We do not require functions $u \in W^{1,2}(D)$ to be continuous - and assume even less we that functions $u \in W^{1,2}(D)$ have a continuous extension to $\overline{D}$. It is therefore not obvious that we can talk about boundary values for functions in $W^{1,2}(D)$. Consider the simple example $\cos(1/x) \in C(0,1)$ which is continuous, but have no continuous extension to $[0,1)$ - wherefore we can not in any meaningful way ascribe a boundary value to $\cos(1/x)$. In order to solve the Dirichlet problem we need the function to take a prescribed boundary value; and therefore we need to have a notion of boundary values. The *"boundary values"* are given by the trace operator T whose existence is assured by the Theorem.

Sketch of the proof of Theorem 2.8: This proof actually goes beyond this course. But I want to indicate how the Trace Theorem is proved for several reasons. First of all we need the theorem. Secondly, in proving the theorem we will encounter some standard techniques in PDE theory. Thirdly, the proof will also indicate why there is more to Sobolev spaces than an integration by parts formula. In particular, I want to stress that measure theory (as in *"Advanced Real Analysis"*) is important for the general analysis of PDE.

Step 1 [Straightening of the boundary.]: *It is enough to prove that the operator* $T : W^{1,2}(B_1^+(0)) \mapsto L^2(B_{3/4}(0) \cap \{x_n = 0\})$*, where* $B_1^+(0) = B_1(0) \cap \{x_n > 0\}$ exists.

Proof of Step 1: By definition a domain has C^1 boundary if we can cover $\partial B_1(0)$ by finitely many balls $B_{r_j}(x^j)$ such that $\partial D \cap B_{2r_j}(x^j)$ is the graph of a C^1 function f^j in some coordinate system.

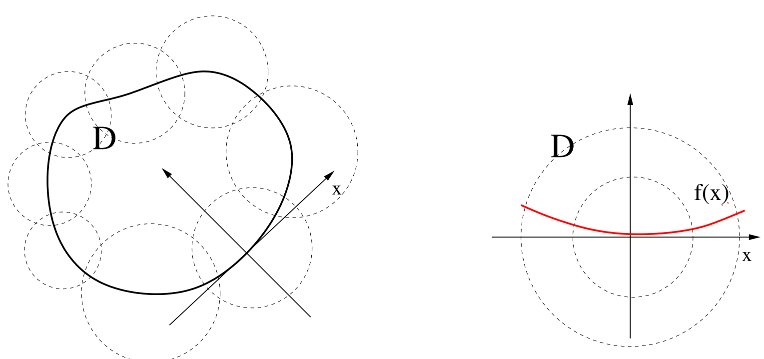


Figure: *The left figure shows a domain D with C^1 boundary. This means that we may cover ∂D by a finite number of balls $B_{r_j}(x^j)$ such that for each ball there is a coordinate system so that $\partial D \cap B_{2r_j}(x^j)$ is a graph in the coordinate system. The right picture shows the same coordinate system rotated and the boundary portion $\partial D \cap B_{2r_j}(x^j)$ (in red) which is clearly the graph of some function $f(x)$. We will change coordinates to straighten the red part of the boundary.*

The idea of the proof is that we may *"straighten the boundary"* in $B_{2r_j}(x)$ by defining the new coordinates $\hat{x}$ so that

$$(x_1, x_2, ..., x_n) = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n + f^j(\hat{x}')) \Leftrightarrow (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n) = (x_1, x_2, ..., x_n - f^j(x')).$$

Then the part of the boundary $x_n = f^j(x')$ will be mapped to the hyperplane $\hat{x}_n = 0$ in the $\hat{x}$ coordinates. We may write the function $u(x)$ in these coordinates as $\hat{u}(\hat{x})$.

By the chain rule we get that

$$\frac{\partial \hat{u}(\hat{x})}{\partial \hat{x}_i} = \sum_{k=1}^{n} \frac{\partial x_k}{\partial \hat{x}_i} \frac{\partial u(x)}{\partial x_k} = \sum_{k=1}^{n} \left(\frac{\partial u(x)}{\partial x_k} + \frac{\partial f^j(x')}{\partial x_k} \frac{\partial u}{\partial x_n} \right).$$

In particular, $|\nabla \hat{u}(\hat{x})|$ will be comparable in size with $|\nabla u(x)|$.

Since $f(x)$ is continuously differentiable we can conclude that

$$\int_{D \cap B_{2r}(x^0)} |\nabla \hat{u}(\hat{x})|^2 d\hat{x} \leq C \int_{D \cap B_{2r}(x^0)} |\nabla u(x)|^2 dx,$$

where the constant C only depend on the maximum value of $|\nabla' f(x')|$.

Notice that $\hat{u}(\hat{x})$ is defined in a set where part of the boundary is straight (in the $\hat{x}$ coordinates). If we can define boundary values for $\hat{u}$ on the straight part of the boundary then we can define boundary values of $u(x)$ on the portion of the boundary that lays in $B_{r_j}(x^j)$ by the equality $u(x) = \hat{u}(x', 0)$. But the entire boundary ∂D can be covered by finitely many balls $R_{r_j}(x^j)$ so we can define boundary values for $u(x)$ on the entire boundary ∂D.

Step 2: *Let $u(x) \in W^{1,2}(B_2^+(0))$ then*

$$\int_{B_1'(0)} |u(x', t) - u(x', s)|^2 dx' \leq |s - t| \|\nabla u(x)\|^2_{L^2(B_2^+)}$$

where $B_1'(0) = \{x' \in \mathbb{R}^{n-1}; |x'| \leq 1\}$ is the unit ball in the x' coordinates.

Proof of step 2: Using the fundamental theorem of calculus[16]

$$\int_{B_1'(0)} |u(x', t) - u(x', s)|^2 dx' = \int_{B_1'(0)} \left| \int_s^t \frac{\partial u(x)}{\partial x_n} dx_n \right|^2 dx' \leq$$

$$\leq |s - t| \int_{B_1'(0)} \int_s^t \left| \frac{\partial u(x)}{\partial x_n} \right| dx' dx_n \leq |s - t| \|\nabla u(x)\|^2_{L^2(B_2^+)}$$

where we used the Cauchy-Schwartz inequality (12)[17] in the first inequality.

[16] This is just a sketch of a proof. But this is the part of the proof that is most sketchy. We have note proved, nor will we prove, that the fundamental theorem of calculus is applicable to functions in $W^{1,2}$ in the way we use it here.

[17] With $g(x) = \frac{\partial u(x)}{\partial x_n}$ and $h(x) = 1$.

Step 3: *Let $u(x) \in W^{1,2}(B_2^+(0))$ then the limit $\lim_{t\to 0'} u(x',t) = u^0(x',0)$ exists (and is therefore unique) and the function $u^0(x',0) \in L^2(B_1'(0))$ and satsifies the estimate*

$$\|u^0\|_{L^2(B_1'(0))} \leq C\|u\|_{W^{1,2}(B_2^+(0))}$$

where the constant C does not depend on u. This finishes the proof.

Proof of Step 3: Since $u(x) \in W^{1,2}(B_2^+(0))$ it follows that

$$\int_0^{1/4}\int_{B_1'(0)} |u(x)|^2 dx' dx_n \leq \int_{B_2^+(0)} |u(x)|^2 dx < \infty, \tag{23}$$

where we used $B_1'(0)\times(0,1/4) \subset B_2^+(0)$ in the first inequality and the definition of $W^{1,2}(B_2^+(0))$ (see Definition 2.4) in the second inequality.

From (23) we can conclude that there exists an $s \in (0,1/4)$ such that

$$\int_{B_1'(0)} |u(x',s)|^2 dx' \leq 4\int_{B_2^+(0)} |u(x)|^2 dx.$$

This implies that $u(x',s) \in L^2(B_1'(0))$ and therefore, from step 2, that $u(x',t) \in L^2(B_1'(0))$.

By Step 2 the sequence of functions $u(x',s/j)$ will form a Cauchy sequence and is therefore convergent, in $L^2(B_1'(0))$ to some function $u^0(x',0) \in L^2(B_1'(0))$. Also, by step 2,

$$\|u(x',s/j) - u^0(x',0)\|_{L^2(B_1'(0))} \leq \sqrt{s/j}\|\nabla u(x)\|_{L^2(B_2^+)}.$$

We only need to assure that $\lim_{t\to 0^+} u(x',t) = u^0(x',0)$. But that follows from step 2 and the triangle inequality:

$$\|u(x',t) - u^0(x',0)\|_{L^2(B_1'(0))} \leq \tag{24}$$

$$\leq \|u(x',t) - u^0(x',s/j)\|_{L^2(B_1'(0))} + \|u(x',s/j) - u^0(x',0)\|_{L^2(B_1'(0))} \leq \tag{25}$$

$$\left(\sqrt{t-s/j} + \sqrt{s/j}\right)\|\nabla u(x)\|_{L^2(B_2^+)}, \tag{26}$$

where we choose j so large that $s/j \leq t$. It clearly follows from (24)-(26) that

$$\|u(x',t) - u^0(x',0)\|_{L^2(B_1'(0))} \leq 2\sqrt{t}\|\nabla u(x)\|_{L^2(B_2^+)}. \tag{27}$$

It follows that $\lim_{t\to 0'} u(x',t) = u^0(x',0)$ in $L^2(B_1'(0))$. □

We also need two more results on Sobolev spaces and traces.

Corollary 2.1. *Let D be a bounded C^1 domain and $u \in W^{1,2}(D)$ and $u = f$ on ∂D in the sense of traces. Then*

$$\|u\|_{L^2(D)} \leq C\left(\|\nabla u\|_{L^2(D)} + \|f\|_{L^2(\partial D)}\right), \tag{28}$$

where the constant C does not depend on u.

Sketch of the Proof: We will only sketch the proof. We begin with the special case $f(x) = 0$. Since D is bounded there is a cube $Q_R = \{x \in \mathbb{R}^n;\ |x_i| < R$ for $i = 1, 2, ..., n\}$ and $D \subset Q_R$. We will extend u to zero in $Q_R \setminus D$:

$$u(x) = \begin{cases} u(x) & \text{for } x \in D \\ 0 & \text{for } x \in Q_R \setminus D. \end{cases}$$

By the fundamental theorem of calculus

$$u(x', x_n) = \int_{-R}^{x_n} \frac{\partial u(x', t)}{\partial x_n} dt. \tag{29}$$

If we take absolute values and square both sides of (29) and then integrate over D we get

$$\int_D |u(x)|^2 dx = \int_D \left| \int_{-R}^{x_n} \frac{\partial u(x',t)}{\partial x_n} dt \right|^2 dx \leq$$

$$\leq 2R \int_D \int_{-R}^{x_n} \left| \frac{\partial u(x',t)}{\partial x_n} \right|^2 dt dx \leq 2R \int_D \int_{-R}^{x_n} \left| \nabla u(x',t) \right|^2 dt dx,$$

where we used the Cauchy-Schwartz inequality[18] in the first inequality. Changing the order of integration leads to

$$\int_D |u(x)|^2 dx \leq 2R \int_{-R}^{R} \int_D \left| \nabla u(x', x_n) \right|^2 dx dt \leq 4R^2 \int_D \left| \nabla u(x', x_n) \right|^2 dx,$$

where we also increased the interval of integration from $-R < t < x_n$ to $-R < t < R$ which clearly increases the value of the integral. This proves (28) in the case $f(x) = 0$.

For the general case we may define a cut-off function $\psi(x) \in C_c^\infty(D)$ such that $0 \leq \psi(x) \leq 1$ and $\psi(x) = 1$ for $\mathrm{dist}(x, \partial D) > r_0/2$ where r_0 is the smallest radius in the proof of the Trace Theorem. We then split up $u(x) = (1 - \psi(x))u(x) + \psi(x)u(x)$. Then $\psi(x)u(x)$ has trace equal to zero on ∂D and the argument in the previous paragraph applies. The function $(1 - \psi(x))u(x)$ can be estimated in terms of the boundary values and the norm $\|\nabla(1 - \psi)u\|_{L^2}$ as in (27). We leave the details to the reader. □

The final corollary we state without proof.

Corollary 2.2. *Let $u^j \rightharpoonup u^0$ in $W^{1,2}(D)$ where D is a bounded C^1 domain. Assume furthermore that $u^j = f$ on ∂D in the trace sense. Then $u^0 = f$ on ∂D.*

Exercises.

1. Let D be a bounded domain and $u(x) \in C^1(D)$. Show that $u(x) \in W^{1,2}(D)$.

[18] Theorem 2.6 with $g(x) = \frac{\partial u(x',t)}{\partial x_n}$ and $h(x) = 1$.

2. * Consider the function $u(x) = \ln(1/|x|)$ defined in $B_1(0) \subset \mathbb{R}^3$.

   (a) Show that $u(x) \in W^{1,2}(B_1(0))$.

   (b) Conclude that there are discontinuous, and even unbounded, functions $u(x) \in W^{1,2}(B_1(0))$.

3. ** Let D be a bounded domain and $u(x) \in C^1(D)$.

   (a) If $D = B_1(0)$ and $u(x) = 0$ on $\partial B_1(0)$ show that

   $$u(0) = \frac{1}{\omega_n} \int_{B_1(0)} \frac{y \cdot \nabla u(y)}{|y|^n} dy.$$

   HINT: *By the fundamental theorem of calculus* $u(x) = \int_0^1 y \cdot \nabla u(ty)dt$ *for any* y *such that* $|y| = 1$. *Integrate this over the unit sphere* $\partial B_1(0) = \{y;\ |y = 1|\}$.

   (b) Show that for all $x \in B_1(0)$.

   $$u(x) = \frac{1}{\omega_n} \int_{B_1(0)} \frac{(y - x) \cdot \nabla u(y)}{|y - x|^n} dy.$$

   (c) Use the following inequality, known as Hölder's inequality,

   $$\int_{B_1} f(x)g(x)dx \leq \left(\int_{B_1} |f(x)|^p dx \right)^{1/p} \left(\int_{B_1} |f(x)|^q dx \right)^{1/q}$$

   for $\frac{1}{p} + \frac{1}{q} = 1$, to show that for any $\epsilon > 0$ there exists a constant C_ϵ such that

   $$\sup_{B_1(0)} |u(x)| \leq C_\epsilon \left(\int_{B_1(0)} |\nabla u(x)|^{n+\epsilon} \right)^{\frac{1}{n+\epsilon}}.$$

   [REMARK:] *The assumption that* $u \in C^1$ *is not needed in the above argument. It is enough to assume that* $u \in W^{1,n+\epsilon}(B_1(0))$. *The exercise therefore shows that any function in* $W^{1,p}(B_1(0))$ *that vanishes on* $\partial B_1(0)$*is bounded.*

4. Assume that $u^j(x) \rightharpoonup u^0(x)$ in $L^2(-\pi, \pi)$. Show that all the Fourier coefficients of u^j converges to the corresponding Fourier coefficients of u^0.

5. * Show that the weak derivatives of the following functions, $f(x)$, either exist or does not exist. Then calculate the weak derivative.

   (a) $f(x) = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x < 0 \end{cases}$ where $f(x)$ is defined on $[-1, 1] \subset \mathbb{R}$.

   Does weak derivatives have to be continuous?

(b) $f(x) = \begin{cases} \frac{1}{x^{1/4}} \text{ if } x > 0 \\ 0 \text{ if } x < 0 \end{cases}$ where $f(x)$ is defined on $[-1,1] \subset \mathbb{R}$.

(c) $f(x) = \begin{cases} x^{3/4} \text{ if } x > 0 \\ 0 \text{ if } x < 0 \end{cases}$ where $f(x)$ is defined on $[-1,1] \subset \mathbb{R}$.

Does weak derivatives have to be bounded?

(d) $f(x) = \begin{cases} \frac{x_1}{|x|} \text{ if } |x| > 0 \\ 0 \text{ if } x = 0 \end{cases}$ where $f(x)$ is defined on $B_1 \subset \mathbb{R}^3$.

6. * [The need for traces.] The following exercise is meant to shed light on the trace theorem.

   (a) Let $u = \cos(\ln(|x|))$ be a function defined on $(0,1)$. Does $u(x)$ satisfy the assumptions of the trace Theorem? Can you think of any meaningful way to assign a boundary value of $u(x)$ at $x = 0$?[19] and $u = \cos(1/|x|)$ in $\mathbb{R}^n$.

   (b) Let $u(x) = \cos(\ln|x|)$ be a function defined on $B_1 \setminus \{0\}$ in $\mathbb{R}^3$. Show that the function $u(x) \in W^{1,2}(B_1 \setminus \{0\})$. However, there is no any meaningful way to ascribe a boundary value of $u(x)$ to the boundary point $x = 0$. Note that the boundary is not the graph of a C^1-function at $x = 0$.

7. ** [A really bad function.] In this exercise we will construct a really bad function - in mathematical analysis we love bad functions as examples.

   (a) Show that $u(x) = \begin{cases} \frac{x_1}{|x|^{4/3}} \text{ if } |x| > 0 \\ 0 \text{ if } x = 0 \end{cases}$ satisfies $u(x) \in W^{1,2}(B_2(0))$ when the space dimension $n \geq 3$. Also show that $u(x)$ is not bounded in any neighborhood of $x = 0$.

   (b) Since $\mathbb{Q}^3$ is countable we may define a sequence $\{q_j\}_{j=1}^{\infty}$ such that $\cup_{j=1}^{\infty}\{q_j\} = \mathbb{Q}^3 \cap B_1^+(0)$. Define $w(x) = \sum_{j=1}^{\infty} 2^{-j} u(x - q_j)$ and show that $w(x) \in W^{1,2}(B_1^+(0))$ and that $w(x)$ is not bounded, neither from above nor from below, on any open set of $B_1^+(0)$.

   [Hint:] *In order to show that $w(x) \in W^{1,2}(B_1^+(0))$ it might be helpful to use the following triangle inequality* $\left\| \sum_j f_j(x) \right\|_{W^{1,2}} \leq \sum_j \|f_j(x)\|_{W^{1,2}}$.

   (c) What is $\limsup_{x \to x^0} w(x)$ and $\liminf_{x \to x^0} w(x)$ for $x^0 \in B_1(0) \cap \{x_n = 0\}$?

   (d) Does this weird function have well defined boundary values, in the trace sense, on $B_{1/2}(0) \cap \{x_n = 0\}$?

[19] The point of the second question is that you should realize that it is not obvious for a function to have boundary values.

3 The Obstacle Problem.

In this section we will consider *the obstacle problem*. The obstacle problem consists of to minimizing

$$J(u) = \int_D F(\nabla u(x))dx = \int_D |\nabla u(x)|^2 dx, \tag{30}$$

in the set

$$K = \{u \in W^{1,2}(D);\ u(x) = f(x) \text{ on } \partial D \textbf{ and } u(x) \geq g(x) \textbf{ in } D\}. \tag{31}$$

Notice that the set K is the convex set of all functions $u(x)$ that achieves the boundary data $f(x)$ (in the trace sense) and $u(x) \geq g(x)$ in the domain D. The difference between the obstacle problem and the minimization in Theorem 2.3 is that in the obstacle problem we require that the graph of the minimizer should stay above a prescribed obstacle $g(x)$.

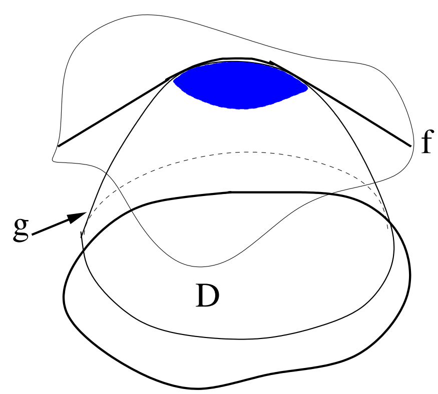


Figure 5: *The graph of a typical solution to the obstacle problem.*

The obstacle problem is much more complicated than the normal Dirichlet problem. For instance, the obstacle problem is non-linear and the obstacle problem has a new unknown: the set where $u(x) = g(x)$ or equivalently the set $\Omega = \{x;\ u(x) > g(x)\}$. Of particular importance is the boundary of the set $\overline{\Omega}$, we call this boundary *The Free Boundary* and denote it $\Gamma = \partial\overline{\Omega}$.

We will denote

$$\Omega = \{x \in D;\ u(x) > g(x)\},\ \Gamma = \partial\overline{\Omega} \text{ (=Free Boundary)}.$$

We are interested in existence of solutions and properties of the solutions and in particular the properties of the sets Ω and Γ.

3.1 Existence of Solutions and some other questions.

To prove that a solution exists is a standard application of Theorem 2.2. In particular we have the following theorem.

Theorem 3.1. *Let D be a bounded domain with C^1 boundary, $f(x) \in C(\partial D)$ and $g(x) \in W^{1,2}(D)$. Assume furthermore that the set*

$$K = \{u \in W^{1,2}(D);\ u(x) = f(x) \text{ on } \partial D \textbf{ and } u(x) \geq g(x) \textbf{ in } D\}.$$

is non-empty. Then there exists a unique function $u(x) \in K$ such that

$$J(u) = \int_D |\nabla u(x)|^2 dx \leq \int_D |\nabla v(x)|^2 dx$$

for all $v(x) \in K$.

Proof: This is proved in the same way as Theorem 2.2. If u^j is a minimizing sequence[20] then $\|\nabla u^j\|_{L^2(D)}$ is bounded and, by Corollary 2.1 $\|u^j\|_{W^{1,2}(D)}$ is bounded. We may thus find a sub-sequence $u^k \rightharpoonup u$ in $W^{1,2}(D)$. By Corollary 2.2 $u^0 = f$ on ∂D and thus $u \in K$. By convexity of the functional $J(u)$ it follows, as in Theorem 2.2, that $J(u) \leq \lim_{k\to\infty} J(u^{j_k}) = \inf_{v\in K} J(v)$. □

We have now entirely left the nice and comfortable kind of mathematics where we can explicitly calculate our solutions. Theorem 3.1 is an abstract existence theorem and does not indicate how we should even begin to calculate the minimizer $u(x)$. In general, even for rather nice domains D and functions $f(x)$ and $g(x)$, we have no idea how to calculate the value of $u(x)$. We have, however, a minimizer $u(x)$ and that minimizer is unique and we would like to describe this minimizer as completely as possible. The questions we will ask are:

1. Does the minimizer $u(x)$ of the obstacle problem satisfy a partial differential equation (as the minimizer to the Dirichlet energy in Theorem 2.3 did.)? We usually call the PDE that the minimizer solves for the Euler-Lagrange equation.

2. Does the minimizer of the obstacle problem satisfy any other "good" properties? Is the minimizer continuous, differentiable or even analytic?

3. What can be said about the set Ω? Is Ω an open set? Is the boundary differentiable? Is there anything that characterize the boundary?[21]

The only thing we know about $u(x)$ is that $u(x) \in K$ and that $u(x)$ is a minimizer of the Dirichlet energy among all functions in K. Therefore, we need to start our investigation with an investigation into what it means to be a minimizer - what variations can we do and what can these variations tell us about the solution.

[20] That is a sequence $u^j \in K$ such that $J(u^j) \to \inf_{v\in K} J(v)$.

[21] In these notes we will not discuss the differentiability properties of Γ.

Variations. If $u(x)$ is a minimizer of $J(u)$ in a convex set K (such as the obstacle problem) and $v \in K$ then, by convexity of K, $(1-t)u + tv \in K$ for all $t \in [0,1]$. Therefore, since $u(x)$ is a minimizer,

$$\int_D |\nabla u(x)|^2 dx \le \int_D |\nabla((1-t)u + tv)|^2 \, dx \tag{32}$$

$$\Rightarrow \int_D \left(2t\nabla u \cdot \nabla(v-u) + t^2|\nabla v|^2\right) \ge 0 \Rightarrow \int_D \nabla u \cdot \nabla(v-u) \ge 0., \tag{33}$$

where we divided by $t > 0$ and then sent $t \to 0$ in the last implication.

We can therefore conclude that[22]

$$\text{If } v(x) \in K \text{ then } \int_D \nabla u \cdot \nabla(v-u) \ge 0. \tag{34}$$

Furthermore, if $v(x) \in K$ happens to be a function such that $(1-t)u+tv \in K$ for all $t \in (-\epsilon, \epsilon)$ then

$$\int_D \nabla u \cdot \nabla(v-u) = 0. \tag{35}$$

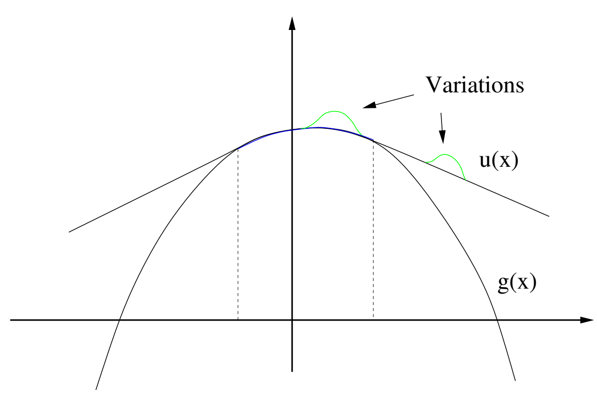
This can easily be seen by replicating the argument in (32)-(33) and using that the inequality reverses direction for $t < 0$.

If we choose the variation $v(x) = u(x) + \phi(x)$, for any $\phi \ge 0$ and $\phi \in W^{1,2}(D)$ with compact support, in (34) then we get

$$\int_D \nabla u \cdot \nabla \phi \ge 0. \tag{36}$$

And if $\mathrm{spt}(\phi) \subset \{u(x) > g(x)\}$ then we actually get, from (35), that

$$\int_D \nabla u(x) \cdot \nabla \phi(x) = 0. \tag{37}$$



[22] This condition is usually called *"a variational inequality"*.

Figure 6: *A graphical representation of the varations, if the bump ϕ is added in a region where $u(x) = g(x)$ (as in the left bump in the graph) then we get the inequality (36). But if the bump is added away from the touching set (as the right bump) then we may do variations with $t \in (-\epsilon, \epsilon)$ and thus get the full equality as in (37).*

An informal argument and the way ahead: We would like to make an integration by parts in (36) in order to deduce that

$$0 \leq \int_D \nabla u \cdot \nabla \phi = \left\{ \begin{array}{l} \text{unjustified int.} \\ \text{by parts} \end{array} \right\} = -\int_D \phi \Delta u, \tag{38}$$

Heuristically, since $\phi \geq 0$, the calculation (38) implies that $\Delta u \leq 0$ in D. Similarly, from (37) we would like to deduce that $\Delta u = 0$ in the set $\{x \in D;\ u(x) > g(x)\}$. Notice that this would directly imply that the equality $u(x) = g(x)$ can only happen whenever $\Delta g(x) \leq 0$. It would thus give us some information about the set Ω (technically about Ω^c).

The problem with the calculation in (38) is that is assumes that $u(x)$ has second derivatives. We must first prove that $u(x)$ has second derivatives (in some sense) in order to justify (38). This indicates that we need to develop a regularity theory for the obstacle problem.[23] Our next goal will be to show that a solution to the obstacle problem has weak second derivatives. But before we can do that we need to make a simplifying assumption and reformulate the problem slightly.

Normalized solutions to the Obstacle problem: If we assume that $g \in C^2(D)$ and define $v(x) = u(x) - g(x) \geq 0$ then $v(x)$ minimizes

$$\int_D |\nabla(v(x) + g(x))|^2 = \int_D |\nabla v(x)|^2 dx + \int_D 2\nabla v(x) \cdot \nabla g(x) + \int_D |\nabla g(x)|^2 dx =$$

$$= 2\int_D \left(\frac{1}{2}|\nabla v|^2 - v\Delta g(x) \right) dx + \int_D |\nabla g(x)|^2 dx + \int_{\partial D} v(x) \frac{\partial g(x)}{\partial \nu} dA,$$

where we used an integration by parts in the last equality. Notice that since $g(x)$ is a given function (independent of $v(x)$) the integral $\int_D |\nabla g(x)|^2 dx$ is independent of $v(x)$. Also, since $u \in K$ which means that $u(x) = f(x)$ on ∂D it follows that $v(x) = u(x) - g(x) = f(x) - g(x)$ on ∂D. In particular,

$$\int_D |\nabla g(x)|^2 dx + \int_{\partial D} v(x) \frac{\partial g(x)}{\partial \nu} dA =$$

$$= \int_D |\nabla g(x)|^2 dx + \int_{\partial D} (f(x) - g(x)) \frac{\partial g(x)}{\partial \nu} dA$$

[23] Regularity theory is the part of PDE theory where one proves that solutions to a PDE are more regular, that is have more derivatives, than what is needed to define the solution. Regularity theory also includes deriving a priori estimates of the norm of the solution.

is just a constant independent of $v(x)$. It follows that $v(x)$ is a minimizer of the energy

$$\int_D \left(\frac{1}{2}|\nabla v|^2 - v\Delta g(x) \right) dx \tag{39}$$

in the set

$$\tilde{K} = \left\{ v \in W^{1,2}(D);\ v(x) \geq 0 \text{ and } v(x) = f(x) - g(x) \text{ on } \partial D \right\}.$$

In many situations it is somewhat easier to work with the function $v(x)$ instead of $u(x)$. It is in particular much easier to work with the formulation with $v(x)$ if $\Delta g(x) = -1$. We therefore make the following definition.

Definition 3.1. *We say that $u(x)$ is a solution to the normalized obstacle problem if $u(x)$ minimizes*

$$\int_D \left(\frac{1}{2}|\nabla u(x)|^2 + u(x) \right) dx \tag{40}$$

among all functions in the set

$$K = \left\{ u \in W^{1,2}(D);\ u(x) \geq 0 \textit{ and } u(x) = f(x) \textit{ on } \partial D \right\}.$$

In the rest of these notes we will study solutions to the normalized obstacle problem.

The normalized obstacle problem is somewhat less general than the general obstacle problem. In particular the assumption that $\Delta g(x) = -1$ is a sever limitation. We are however willing to pay the price of a less general problem in order to get a simpler problem.

Exercises:

1. [Comparison principle.] Let $u(x)$ and $v(x)$ be solutions to the normalized obstacle problem in a domain D. Furthermore assume that $v(x) \geq u(x)$ on ∂D. Prove that $v(x) \geq u(x)$ in D.

   Hint: *Assume the contrary, that $u(x) > v(x)$ in some set Σ, and make a variation with $\phi = \max(u(x) - v(x), 0)$.*

2. $*\frac{*}{2}$ Let $u(x)$ and $v(x)$ be as in the previous exercise and assume furthermore that $v(x) > u(x)$ on part of the boundary ∂D and that D is connected. Does it follow that $v(x) > u(x)$ in the entire domain D? Would your answer be the same if $u(x)$ and $v(x)$ where harmonic functions?

3. * Let $u(x)$ and $v(x)$ be solutions to the obstacle problem in D with obstacles $g_u(x)$ and $g_v(x)$ respectively. Assume that $u(x) = v(x)$ on ∂D and prove that if $g_v(x) \geq g_u(x)$ in D then $v(x) \geq u(x)$ in D.

4 Regularity Theory.

4.1 Solutions to the Normalized Obstacle Problem has second derivatives

In this section we will show that a solution to the normalized obstacle problem has weak second derivatives. We begin with a *difference quotient argument*, this is a standard argument in PDE theory and the calculus of variations. In the proof we use the notation $e_1 = (1, 0, 0, .., 0), \cdots, e_i = (0, .., 0, 1, 0, ..)$ for the standard unit vectors.

Lemma 4.1. *Let $u(x)$ be a solution to the normalized obstacle problem in a domain D. Then for each compact set $\mathcal{C} \subset D$ there exists a constant C only depending on dist$(\mathcal{C}, D^c)$ such that*

$$\int_{\mathcal{C}} \left| \frac{\nabla(u(x+e_ih) - u(x))}{h} \right|^2 dx \le C \int_D \left| \frac{u(x+e_ih) - u(x)}{h} \right|^2 dx \tag{41}$$

for any $h \in \mathbb{R}$ satisfying $|h| < dist(\mathcal{C}, D^c)/2$.

Proof: We know that

$$0 \le \int_D (\nabla u(x) \cdot \nabla \phi(x) + \phi(x))\, dx \tag{42}$$

for any ϕ with compact support such that $v + t\phi \in K$, that is if $v + t\phi \in K \ge 0$, for $t \in (0, \epsilon)$.

Now we choose $\phi(x) = \psi(x)^2(u(x + e_ih) - u(x))$ for some $\psi \in C_c^\infty(D)$ that satisfies

1. $0 \le \psi(x) \le 1$ for all $x \in D$,
2. $\psi(x) = 1$ for $x \in \mathcal{C}$,
3. $\psi(x) = 0$ for all x such that $\mathrm{dist}(x, \mathcal{C}) > \frac{\mathrm{dist}(\mathcal{C}, D^c)}{2}$ and
4. $|\nabla\psi(x)| < \frac{4}{\mathrm{dist}(\mathcal{C}, D^c)}$.

Notice that for any $t \in [0, 1)$ we have

$$u(x) + t\phi(x) = t\psi^2(x)u(x + e_ih) + (1 - \psi^2(x))u(x) \ge 0,$$

since $u(x) \ge 0$. Also $u(x) + t\psi^2(x)(u(x + e_ih) - u(x)) = f(x)$ on ∂D since $\psi(x) = 0$ on ∂D and $u(x) = f(x)$ on ∂D.[24]

[24] There is a slight technical detail that should be mentioned here. Since $u(x)$ is defined on D it follows that $u(x+he_i)$ is defined on the set $D_{-h} = \{x;\ x+he_i \in D\} \ne D$. In particular, the function $u(x) + t\psi(x)^2(u(x + e_ih) - u(x))$ is only defined on $D_{-h} \cap D$ which is a strictly smaller set than D. But since $\psi(x) = 0$ on $D \setminus (D_{-h} \cap D)$ for $|h| < \mathrm{dist}(\mathcal{C}, D^c)/2$ we may consider the function that equals $u(x)+t\psi(x)^2(u(x+e_ih)-u(x))$ in $D_{-h} \cap D$ and equals zero in $D \setminus (D_{-h} \cap D)$ for $|h| < \mathrm{dist}(\mathcal{C}, D^c)/2$. That function is well defined and all the calculations goes through for that function. It is not uncommon that one uses the simplified convention that an undefined function times zero is zero - it simplifies things.

With this choice of $\phi(x)$ in (42) we arrive at

$$\int_D \Big(\nabla u(x)\cdot\nabla(\psi(x)^2(u(x+e_ih)-u(x))) + (\psi(x)^2(u(x+e_ih)-u(x)))\Big)dx \geq 0. \tag{43}$$

Next we notice that $u(x+he_i)$ is a minimizer if the normalized obstacle problem in a slightly shifted domain with boundary values $f(x+he_i)$. Arguing similarly as above we arrive at (with $\phi(x) = \psi(x)^2(u(x)-u(x+he_i))$

$$\int_D \Big(\nabla u(x+he_i)\cdot\nabla(\psi(x)^2(u(x)-u(x+he_i)))+(\psi(x)^2(u(x)-u(x+he_i)))\Big)dx \geq 0. \tag{44}$$

If we add (43) and (44) and rearrange the terms we arrive at

$$0 \geq \int_D \left(\nabla(u(x+e_ih)-u(x))\right)\cdot\nabla\left(\psi(x)^2(u(x+e_ih)-u(x))\right)dx =$$

$$= \int_D \Big(\psi(x)^2\left|\nabla(u(x+e_ih)-u(x))\right|^2\Big)dx +$$

$$+\int_D \left(2\psi(x)(u(x+e_ih)-u(x))\nabla\psi(x)\cdot\nabla(u(x+e_ih)-u(x))\right)dx$$

That is

$$\int_D \psi(x)^2\left|\nabla(u(x+e_ih)-u(x))\right|^2 dx \leq \tag{45}$$

$$\leq -\int_D 2\psi(x)(u(x+e_ih)-u(x))\nabla\psi(x)\cdot\nabla(u(x+e_ih)-u(x))dx.$$

In order to continue we use that for any vectors $\mathbf{v},\mathbf{w}\in\mathbb{R}^n$ we have the following inequality $2\mathbf{v}\cdot\mathbf{w}\leq 2|\mathbf{v}|^2+\frac{1}{2}|\mathbf{w}|^2$ which implies that

$$2\psi(x)(u(x+e_ih)-u(x))\nabla\psi(x)\cdot\nabla(u(x+e_ih)-u(x)) =$$

$$= \underbrace{(2(u(x+e_ih)-u(x))\nabla\psi(x))}_{=\mathbf{v}}\cdot\underbrace{(\psi(x)\nabla(u(x+e_ih)-u(x)))}_{=\mathbf{w}} \leq$$

$$\leq 8|\nabla\psi(x)|^2(u(x+e_ih)-u(x))^2+\frac{1}{2}|\psi(x)|^2\left|\nabla(u(x+e_ih)-u(x))\right|^2 - .$$

Using this in (45) we can deduce that

$$\int_D \psi(x)^2\left|\nabla(u(x+e_ih)-u(x))\right|^2 dx \leq 16\int_D |\nabla\psi(x)|^2(u(x+e_ih)-u(x))^2 dx. \tag{46}$$

Since $\psi(x)=1$ in $\mathcal{C}$ we can estimate the left side of (46) according to

$$\int_{\mathcal{C}} \left|\nabla(u(x+e_ih)-u(x))\right|^2 dx \leq \int_D \psi(x)^2\left|\nabla(u(x+e_ih)-u(x))\right|^2 dx \tag{47}$$

and using that $|\nabla\psi| \le \frac{4}{\mathrm{dist}(\mathcal{C},D^c)}$ we can estimate the right side of (46) according to

$$16\int_D |\nabla\psi(x)|^2(u(x+e_ih)-u(x))^2dx \le \tag{48}$$
$$\le \frac{256}{\mathrm{dist}(\mathcal{C},D^c)^2}\int_D (u(x+e_ih)-u(x))^2dx.$$

Putting (46), (47) and (48) and dividing by h^2 we arrive at (41). □

Lemma 4.1 provides an integral estimate for the difference quotient of the derivatives. But unless we can also show that the right side in (41) is uniformly bounded in h the Lemma would not be very useful. We therefore need the following integral version of the mean value property for the derivatives.

Lemma 4.2. *Assume that $u \in W^{1,2}(D)$ and $\mathcal{C} \subset D$ is a compact set. Then there exists a constant C depending only on the dimension such that for any $|h| \le dist(\mathcal{C}, D^c)$*

$$\int_{\mathcal{C}} \left|\frac{u(x+e_ih)-u(x)}{h}\right|^2 dx \le C\int_\Sigma \left|\frac{\partial u(x)}{\partial x_i}\right|^2 dx. \tag{49}$$

Proof: We will use the following simple version of the Cauchy-Schwartz inequality: *Let $f \in L^2(\Sigma)$ and $|\Sigma|$ denote the area of Σ then*

$$\left|\int_\Sigma |f(x)|dx\right|^2 \le |\Sigma|^{1/2}\int_D |f(x)|^2dx. \tag{50}$$

From the fundamental Theorem of calculus[25] we see that

$$\frac{u(x+e_ih)-u(x)}{h} = \frac{1}{h}\int_0^h \frac{\partial u(x+se_i)}{\partial x_i}ds.$$

Thus

$$\int_{\mathcal{C}} \left|\frac{u(x+e_ih)-u(x)}{h}\right|^2 dx = \int_{\mathcal{C}} \left|\frac{1}{h}\int_0^h \frac{\partial u(x+se_i)}{\partial x_i}ds\right|^2 dx \le \tag{51}$$

$$\le \int_{\mathcal{C}} \frac{1}{h}\int_0^h \left|\frac{\partial u(x+se_i)}{\partial x_i}\right|^2 dsdx, \tag{52}$$

where we used (50), with $\Sigma = (0,h)$, in the last inequality. We may continue to estimate (52) by using the Fubini Theorem

$$\int_{\mathcal{C}} \frac{1}{h}\int_0^h \left|\frac{\partial u(x+se_i)}{\partial x_i}\right|^2 dsdx = \frac{1}{h}\int_0^h\int_{\mathcal{C}} \left|\frac{\partial u(x+se_i)}{\partial x_i}\right|^2 dxds \le \tag{53}$$

$$\le \frac{1}{h}\int_0^h\int_D \left|\frac{\partial u(x)}{\partial x_i}\right|^2 dxds \le \int_D \left|\frac{\partial u(x)}{\partial x_i}\right|^2 dx. \tag{54}$$

[25]Here again we use that the fundamental theorem of calculus holds for Sobolev functions. This is true in an a.e. sense - but we will simply assume it here.

Putting (51), (52), (53) and (54) together gives the Lemma. □

We continue to prove that boundedness of the integral of the difference quotients implies weak differentiability.

Lemma 4.3. *Let $\mathcal{C}\subset D$ be a compact set such that $\tilde{\mathcal{C}}_\delta=\{x;\ dist(x,\mathcal{C})<\delta\}\subset D$.*

Furthermore assume that $u(x)\in L^2(D)$ and that there exists a constant C such that

$$\int_{\tilde{\mathcal{C}}_\delta}\left|\frac{u(x+e_ih)-u(x)}{h}\right|^2 dx\le C, \tag{55}$$

for all $|h|<\delta$.

Then the weak derivative $\frac{\partial u}{\partial x_i}$ exists in $\mathcal{C}$ and

$$\int_{\mathcal{C}}\left|\frac{\partial u(x)}{\partial x_i}\right|^2 dx\le C.$$

Proof: Notice that (55) just states that for any sequence $h_j\to 0$ the functions $\frac{u(x+e_ih)-u(x)}{h}$ are bounded in $L^2(\tilde{\mathcal{C}}_\delta)$. Thus, by the weak compactness theorem for L^2-functions, Theorem 2.7, there exists a sub-sequence, still denoted h_j, such that

$$\frac{u(x+e_ih_j)-u(x)}{h_j}\rightharpoonup g_i(x)\in L^2(\tilde{\mathcal{C}}_\delta).$$

By Lemma 2.1 it follows that $\|g_i\|_{L^2(\tilde{\mathcal{C}}_\delta)}\le C$.

We claim that $g_i(x)$ is the weak x_i-derivative of $u(x)$. To see this we calculate, for any $\phi\in C_c^1(D)$,

$$-\int_{\mathcal{C}}\frac{\partial\phi(x)}{\partial x_i}u(x)dx=\lim_{h_j\to 0}-\int_{\mathcal{C}}\frac{\phi(x+h_je_i)-\phi(x)}{h_j}u(x)dx=$$

$$=\left\{\begin{array}{l}\text{Change of var.}\\ x+h_je_i\to x\\ \text{in } \phi(x+h_je_i)\end{array}\right\}=\lim_{h_j\to 0}\int_{x-e_ih_j\in\mathcal{C}}\phi(x)\frac{u(x)-u(x-e_ih_j)}{h_j}dx\rightharpoonup$$

$$\rightharpoonup\int_{\mathcal{C}}\phi(x)g_i(x)dx.$$

This proves that $g_i(x)=\frac{\partial u(x)}{dx_i}$. □

We are now ready to formulate the statement of this section as a theorem.

Theorem 4.1. *Let $u(x)$ be a solution to the normalized obstacle problem. Then $u(x)$ has weak derivatives of second order on any compact subset $\mathcal{C}$ of D and there exists a constant $C_{\mathcal{C}}$ (depending on $\mathcal{C}$) such that*

$$\int_{\mathcal{C}}\left|D^2u(x)\right|^2 dx\le C_{\mathcal{C}}\int_D|\nabla u(x)|^2dx.$$

Proof: From (41) and (49) we see that

$$\int_{\tilde{\mathcal{C}}_\delta} \left| \frac{\nabla(u(x+e_i h) - u(x))}{h} \right|^2 dx \le C \int_D \left| \frac{\partial u(x)}{\partial x_i} \right|^2 dx, \tag{56}$$

where we have used the notation $\tilde{\mathcal{C}_\delta} = \{x;\ \text{dist}(x,\mathcal{C}) < \delta\} \subset D$ introduced in Lemma 4.3 and chosen $\delta > 0$ small enough so that $\tilde{\mathcal{C}_\delta} \subset D$.

From Lemma 4.3 and (56) we can conclude that $\nabla u(x)$ is weakly differentiable in x_i and

$$\int_{\mathcal{C}} \left| \nabla \frac{\partial u(x)}{\partial x_i} \right|^2 dx \le C_{\mathcal{C}} \int_D \left| \frac{\partial u(x)}{\partial x_i} \right|^2 dx.$$

If we sum this over $i = 1, 2, ..., n$ the theorem follows. □

Exercises:

1. * [Difference Quotients and Regularity Theory.]
   (a) Let $u(x)$ be a minimizer of the Dirichlet energy $\int_D |\nabla u(x)|^2 dx$. Use a difference quotient argument to show that $u \in W^{2,2}(\mathcal{C})$ for any compact set $\mathcal{C} \subset D$.
   (b) Let $u_i(x) = \frac{\partial u(x)}{\partial x_i}$ and show that for any $\psi \in C_c^2(\mathcal{C})$
   $$\int_{\mathcal{C}} \nabla \phi(x) \cdot \nabla u_i(x) dx = 0.$$
   Conclude that u_i is a minimizer to the Dirichlet energy in $\mathcal{C}$.
   (c) Show, by using induction that, $u \in W^{k,2}(\mathcal{C})$ for any $k \in \mathbb{N}$.
2. * Let $g \in W^{1,2}(D)$ and assume that $u(x)$ minimizes
   $$\int_D \left(|\nabla u(x)|^2 + 2u(x)g(x) \right) dx.$$
   Show that $u \in W^{2,2}(D)$.
   Remark: *The same is true for $g \in L^2(D)$, can you prove it?***
3. Verify the change of variables in the proof of Lemma 4.3.
4. * Show that the function
   $$u(x) = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x \le 0 \end{cases}$$
   is not a function in $W^{2,2}(-1,1)$.

4.2 The Euler Lagrange Equations.

Knowing that solutions $u(x)$ to the normalized obstacle problem has second derivatives we are now in position to derive the Euler-Lagrange equations for the obstacle problem. We aim to prove the following theorem.

Theorem 4.2. *Assume D is a C^1-domain and that $u(x)$ minimizes*

$$\int_D \left(\frac{1}{2}|\nabla u(x)|^2 + u(x)\right) dx \tag{57}$$

among all functions in the set

$$K = \left\{u \in W^{1,2}(D);\ u(x) \geq 0 \text{ and } u(x) = f(x) \text{ on } \partial D\right\}.$$

Then

$$\begin{array}{ll} \Delta u(x) = \chi_{\{u(x)>0\}} & \text{in } D \\ u(x) \geq 0 & \text{in } D \\ u \in W^{2,2}_{loc}(D), & \end{array}$$

where

$$\chi_{\{u(x)>0\}} = \begin{cases} 1 & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) < 0. \end{cases}$$

Proof: That $u(x) \in W^{2,2}_{\mathrm{loc}}(D)$ follows from Theorem 4.1. That $u(x) \geq 0$ follows from the fact that $u \in K$. Therefore we only need to show that

$$\Delta u(x) = \chi_{\{u(x)>0\}}. \tag{58}$$

Since $u(x)$ is a minimizer it satisfies the variational inequality

$$\int_D \left(\nabla\phi(x) \cdot \nabla u(x) + \phi(x)\right) dx \geq 0 \tag{59}$$

for all $\phi(x) \geq 0$ such that $\phi(x) \in W^{1,2}_0(D)$ where we used the notation

$$W^{1,2}_0(D) = \{v \in W^{1,2}(D);\ v(x) = 0 \text{ on } \partial D \text{ in the trace sense.}\}$$

Choosing ϕ with compact support in D we may, since $u \in W^{2,2}$, integrate by parts in (59) and derive

$$0 \leq \int_D \left(-\phi(x)\Delta u(x) + \phi(x)\right) dx = \int_D \phi(x)\left(1 - \Delta u(x)\right) dx.$$

Since $\phi(x)$ is arbitrary this already implies that $0 \leq \Delta u(x) \leq 1$. But we claim something stronger, that $\Delta u(x) = \chi_{\{u(x)>0\}}$. In order to derive this we need to make a more refined choice of $\phi(x)$. To that end we choose $\phi(x) = \psi(x)\max(u(x)-\epsilon, 0)$ for some $\psi \in C^1_c(D)$ satisfying $0 \leq \psi(x) \leq 1$. Notice that, with this choice of ϕ it follows that $u(x) + t\phi(x) \in K$ for all $t \in (-\epsilon, \epsilon)$. We can conclude that

$$0 = \int_D \phi(x)\left(1 - \Delta u(x)\right) dx = \int_D \psi(x)\max(u(x)-\epsilon, 0)(1 - \Delta u(x))dx. \tag{60}$$

We will use (60) and a contradiction argument to show that $\Delta u(x) = 1$ in the set $\{u(x) \geq 2\epsilon\}$. Remember that $0 \leq 1 - \Delta u(x) \leq 1$ so if $\Delta u(x) \neq 1$ somewhere in some set $\Sigma \subset \{u(x) \geq 2\epsilon\}$ then $1 - \Delta u(x) < 0$ in that set. If we choose $\psi(x) \geq 0$ to be a function that is strictly positive in (part of) Σ then (60) becomes

$$0 = \int_D \phi(x)\,(1 - \Delta u(x))\,dx = \int_D \underbrace{\psi(x)\max(u(x)-\epsilon,0)}_{\geq 0}\,\underbrace{(1-\Delta u(x))}_{\leq 0}\,dx \leq$$

$$\leq \int_\Sigma \psi(x)\,\underbrace{\max(u(x)-\epsilon,0)}_{\geq \epsilon}\,\underbrace{(1-\Delta u(x))}_{<0}\,dx,$$

this is only possible if Σ has measure zero. We can conclude that $\Delta u(x) = 1$ in $\{u > 2\epsilon\}$ for any $\epsilon > 0$. Sending $\epsilon \to 0$ we can conclude that $\Delta u(x) = 1$ in the set $\{u(x) > 0\}$.

When $u(x) = 0$ then we naturally have $\Delta u(x) = 0$ at almost every point.[26] Therefore

$$\Delta u(x) = \begin{cases} 1 & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) < 0. \end{cases}$$

which is exactly what we wanted to prove. □

Remark: *There is a qurious statement at the end of the proof where we state that $\Delta u(x) = 0$ "at almost every point" of $\{u(x) = 0\}$. The reason for this statement is that, as we will see later, the function*

$$u(x) = \frac{1}{2}(x_n)_+^2 = \begin{cases} \frac{1}{2}x_n^2 & \text{if } x_n > 0 \\ 0 & \text{if } x_n < 0. \end{cases}$$

satisfies $\Delta u(x) = \chi_{\{u(x)>0\}}$. But on the line $x_n = 0$ we have $u(x) = 0$ but $u(x)$ is not even twice differentiable at $x_n = 0$ so $\Delta u(x)$ is not even defined on $\{x_n = 0\}$. Similarly, $u(x) = \frac{1}{2n}|x|^2$ satisfies $\Delta u(x) = 1$ in $\mathbb{R}^n$ so $\Delta u(0) \neq 0$ even though $u(0) = 0$. The almost every means for every x except a set that has zero area. In the above examples the line $\{x_n = 0\}$ and point $\{x = 0\}$ both have finite area and are therefore allowed exceptions. As we already remarked, the theory that we really need in order to make this precise goes beyond this course.

Theorem 4.2 provides us with the Euler-Lagrange equations for solutions to the obstacle problem. In order to continue our investigation of the solutions to the Obstacle problem we would first want to establish that the solutions are continuously differentiable since it is more practical to work with continuously

[26] This result actually needs to be proved (a proof can be found in pretty much any book on Sobolev spaces). But to prove this we need to understand what *almost every point* means. Since we do not want to use to much measure theory in this course we will accept this final statement on faith. But you must admit - it makes you a little curious to see how this is proved.

differentiable functions than with the rather abstract space $W^{2,2}(D)$. We would also like to say something about the free boundary $\Gamma = \partial\{u > 0\}$.

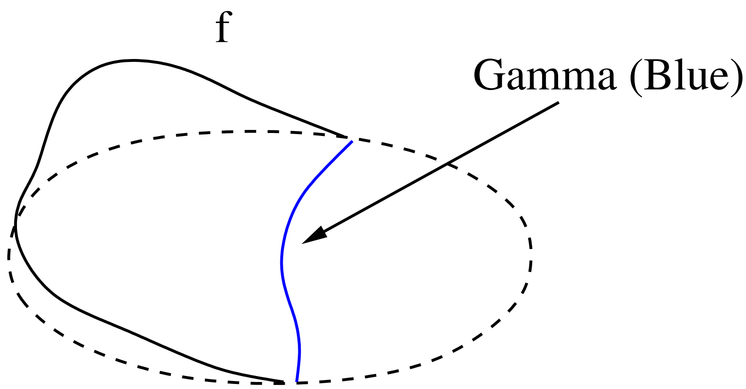


Figure 7: *The graph of a typical solution to the normalized obstacle problem in the ball. The graph takes the values $f(x)$ on the boundary. The solution satisfies $\Delta u(x) = 1$ in part of D and $u(x) = 0$ in the rest of D. In between we have the free boundary Γ (in blue).*

Exercises:

1. * Show that the (normalized) obstacle problem is non-linear. That is prove that if $u(x)$ and $v(x)$ are solutions to the normalized obstacle problem in D. Then it is, in general, not true that $u(x) + v(x)$ is a solution to the normalized obstacle problem.

2. * Show that for any $\alpha > 0$ the function $u(x) = |x|^{-\alpha}$ belongs to $W^{2,2}(B_1)$ if the space dimension n is large enough. Given an α how large must n be?

5 Continuity of the Solution and its Derivatives.

5.1 Heuristics about the free-boundary.

Theorem 4.2 provides the Euler-Lagrange equations for the normalized obstacle problem. But it does not provide any real information on the free boundary $\Gamma_u = \partial\overline{\Omega}_u = \partial\overline{\{x;\ u(x) > 0\}}$. In the next section we will show that the solution to the obstacle problem is a continuously differentiable function which implies that both $u(x) = 0$ and $|\nabla u(x)| = 0$ on Γ_u - this is actually rather strong information on the free boundary itself. In this section we will provide some discussion of the free boundary and try to argue that the Euler-Lagrange equations

$$\begin{array}{ll} \Delta u(x) = \chi_{\{u(x)>0\}} & \text{in } D \\ u(x) \geq 0 & \text{in } D \\ u \in W^{2,2}_{\text{loc}}(D) & \end{array} \tag{61}$$

actually contains some important information about the free boundary.

In applications the free boundary $\Gamma = \partial\overline{\{v > 0\}}$ is often just as important to understand (and calculate) as the solution itself. In particular, the free boundary often describes the boundary of some set of particular importance - such as the region of ice in a melting problem.

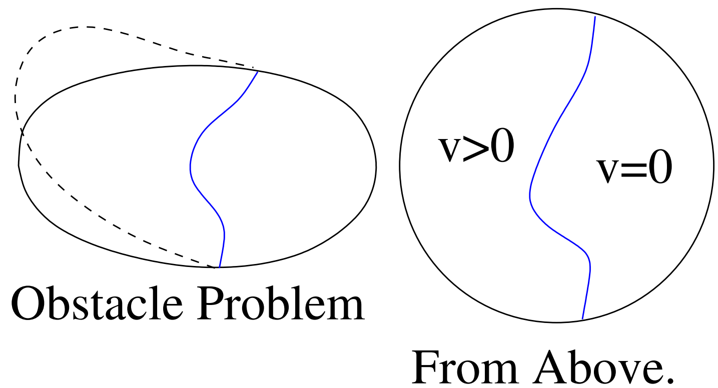


Figure 8: *To the left is the graph of a solution, $v(x)$, to the normalized obstacle problem. In applied problems we are often just as interested in the free boundary Γ, which can be the interface between ice and water in a melting problem. The right picture shows the domain from above with the free boundary marked out. One of the most important questions in free boundary theory is: "Can we describe the free boundary Γ."*

To understand how the free boundary is determined by the Euler-Lagrange equations we we need to understand that The Euler-Lagrange equations (61) is not the same as the solution to

$$\begin{array}{ll} \Delta v(x) = 1 & \text{in the set } \{v(x) > 0\} \\ \Delta v(x) = 0 & \text{in the set } \{v(x) = 0\}. \end{array}$$

The information that $u \in W^{2,2}(D)$ provides extra information that specifies the solution.

To see this we consider the one dimensional example.

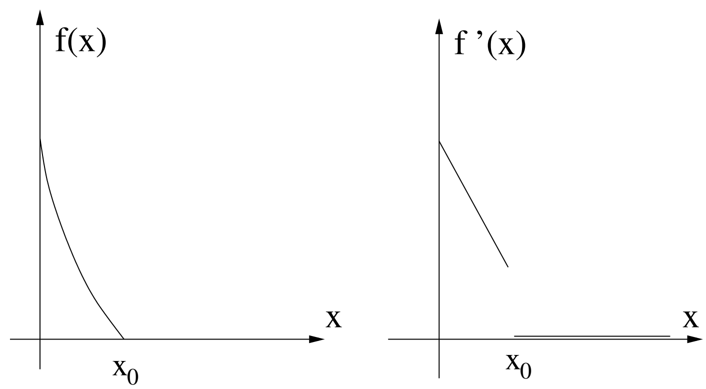
Example: Consider the function

$$f(x) = \begin{cases} \frac{1}{2}(x-1)^2 + (1-x) & \text{for } 0 < x \leq 1 \\ 0 & \text{for } 1 < x < 2. \end{cases}$$

Then

$$\begin{array}{ll} \Delta f(x) = f''(x) = 1 & \text{in the set } \{v(x) > 0\} \\ \Delta f(x) = f''(x) = 0 & \text{in the set } \{v(x) = 0\}, \end{array} \tag{62}$$

but $f \notin W^{2,2}([0,2])$.



Then

Figure 9: *The graph of the function $f(x)$ and its derivative $f'(x)$.*

In particular, for small $h > 0$

$$\int_{1/2}^{3/2} \left| \frac{f'(x+h) - f'(x)}{h} \right|^2 \approx \int_{1-h}^{1} \left| \frac{1}{h} \right|^2 dx = \frac{1}{h} \to \infty, \tag{63}$$

since $f'(x) = \begin{cases} x-2 & \text{if } x < 1 \\ 0 & \text{if } x > 1. \end{cases}$ It follows that the difference quotients $\frac{f'(x+h)-f'(x)}{h}$ does not converge in L^2. The function $f(x)$ is therefore not a solution to the obstacle problem even though it satisfies the equations (62).

From the above example we see that it is not really the following equations that are important:

$$\Delta u(x) = \begin{cases} 1 & \text{in the set } \{u(x) > 0\} \\ 0 & \text{in the set } \{u(x) = 0\}. \end{cases}$$

But the equation

$$\Delta u(x) = \chi_{\{u(x)>0\}},$$

together with the fact that $u \in W^{2,2}(D)$.

The minimization problem *"chooses"* the free boundary $\Gamma = \partial\overline{\{u > 0\}}$ in such way that $u \in W^{2,2}_{\text{loc}}(D)$. From the example above it seems reasonable to conjecture that what determines the position of the free boundary Γ is that the solution should satisfy two boundary conditions, $u(x) = 0$ and $|\nabla u(x)| = 0$ on Γ. The aim of the next section is to prove this.

Exercises:

1. * Find the solution to the following normalized obstacle problem:

$$\text{minimize } \int_{-2}^{2} \left(\frac{1}{2} \left(\frac{\partial u(x)}{\partial x} \right)^2 + u(x) \right) dx$$

in the set

$$K = \left\{ u(x) \in W^{1,2}(-2,2);\ u(-2) = 0,\ u(2) = 2 \text{ and } u(x) \geq 0 \right\}.$$

HINT: *Since this is a one dimensional problem you can calculate it explicitly. Assume that $u > 0$ in $(\gamma, 2]$ and use the equation $\Delta u(x) = 1$ in $(\gamma, 2)$ to calculate $u(x)$. For which γ is $u \geq 0$ satisfies? What is the energy of the function $u(x)$ for a given γ?*

2. Check the calculation (63).

5.2 $C^{1,1}-$estimates for the solution.

In this section we aim to prove that the solution satisfies two boundary conditions on the free boundary Γ.

Remember that the Dirichlet problem:

$$\begin{array}{ll} \Delta u(x) = h(x) & \text{in } D \\ u(x) = f(x) & \text{on } \partial D \end{array}$$

has a unique solution.[27] This means that the boundary data $f(x)$ and the domain D determines the value of $\nabla u(x)$ for every $x \in \partial D$.

For the obstacle problem things are different. The set $\Omega = \{x \in D;\ u(x) > 0\}$ is part of the solution and we might ask what is the criteria that determined the set Ω; or equivalently the free boundary $\Gamma = \partial\overline{\Omega}$. In this section we will prove that the set Ω is the unique set such that:

1. $\Omega \subset D$,

2. $\text{spt}(f) \subset \overline{\Omega}$ and

3. if $u(x)$ solves the Dirichlet problem

$$\begin{array}{ll} \Delta u(x) = 1 & \text{in } \Omega \\ u(x) = f(x) & \text{on } \partial\Omega \cap \partial D \\ u(x) = 0 & \text{on } \partial\Omega \setminus \partial D \end{array}$$

then $|\nabla u| = 0$ on $\partial\Omega \setminus \partial D$ and $u \geq 0$ in Ω.

This means that Ω is a very special set - and an arbitrarily chosen set $\Sigma \subset D$ will not satisfy 3.

We begin our proof with a Lemma about solutions to the Poisson equation.

[27] As a matter of fact, we need some mild extra assumption on the solution in order to proclaim uniqueness. For instance, there exists only one solution $u \in C^2(D)$ that is continuous in $\overline{D}$.

Lemma 5.1. *Let $h(x)$ be a bounded and integrable function, $|h(x)| \leq M$, with support in a bounded set $D \subset \mathbb{R}^n$, $n \geq 3$, and define*

$$u(x) = -\frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{h(z)}{|x-z|^{n-2}} dz, \tag{64}$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$

Then

$$\sup_{x \in \mathbb{R}^n} |u(x)| \leq \frac{M}{2(n-2)} diam(D)^2 \tag{65}$$

and there exists constant C_n, that only depend on the dimension, such that

$$|u(x) - u(y)| \leq C_n diam(D) M |x-y| \tag{66}$$

for any $x, y \in \mathbb{R}^n$. Here diam(D) is the diameter of the smallest ball that contains D.

Remark: *Remember that the function $u(x)$ defined as in (64) satisfies $\Delta u(x) = h(x)$. A proof of this, for $h(x) \in C_c^2(\mathbb{R}^n)$ can be found in Evans. The result is also true for bounded and integrable $h(x)$. It is also true for more general integrable functions $h(x)$.*

In the Lemma we assume that $n \geq 3$. A similar result is also true for $n = 2$. But when $n = 2$ the Newtonian kernel is logarithmic and thus not bounded at infinity. This leads to some small technical differences in the statement of the theorem. For the sake of brevity we will not include the $\mathbb{R}^2$ case in these notes.

Proof: We begin by proving (65). Pick an arbitrary point $x \in \mathbb{R}^n$. Then

$$\begin{aligned} |u(x)| &\leq \frac{1}{(n-2)\omega_n} \left| \int_{\mathbb{R}^n} \frac{h(z)}{|x-z|^{n-2}} dz \right| \leq \\ &\leq \frac{M}{(n-2)\omega_n} \left| \int_{B_{\text{diam}(D)}(x)} \frac{1}{|x-z|^{n-2}} dz \right| \leq \\ &\leq \frac{M}{(n-2)\omega_n} \left| \int_{B_{\text{diam}(D)}(0)} \frac{1}{|z|^{n-2}} dz \right|, \end{aligned} \tag{67}$$

where we translated the coordinates $x - z \mapsto z$ in the last step.

If we change to polar coordinates in (67) we arrive at

$$|u(x)| \leq \frac{M}{(n-2)} \int_0^{\text{diam}(D)} r dr = \frac{M}{2(n-2)} \text{diam}(D)^2$$

Since x was arbitrary this proves (65).

Next we prove (66). To that end we pick two points $x, y \in \mathbb{R}^n$. If $|x-y| \geq \text{diam}(D)$ then (65) implies that

$$|u(x) - u(y)| \leq |u(x)| + |u(y)| \leq \frac{M}{(n-2)} \text{diam}(D)^2 \leq \frac{M}{(n-2)} \text{diam}(D) |x-y|.$$

It is therefore enough to prove (66) for $|x-y| < \operatorname{diam}(D)$. For the rest of the proof we define $r = |x-y|$ and assume, without loss of generality, that $r < \operatorname{diam}(D)$.

From the defining formula of u we can derive

$$\begin{aligned}
|u(x)-u(y)| \le \frac{1}{(n-2)\omega_n}\left|\int_{\mathbb{R}^n}\left(\frac{h(z)}{|x-z|^{n-2}}-\frac{h(z)}{|y-z|^{n-2}}\right)dz\right| \le \\
\le \frac{1}{(n-2)\omega_n}\left|\int_{\mathbb{R}^n\setminus B_{3r}((x+y)/2)}\left(\frac{h(z)}{|x-z|^{n-2}}-\frac{h(z)}{|y-z|^{n-2}}\right)dz\right| + \\
+\frac{1}{(n-2)\omega_n}\left|\int_{B_{3r}((x+y)/2)}\frac{h(z)}{|x-z|^{n-2}}dz\right| + \\
+\frac{1}{(n-2)\omega_n}\left|\int_{B_{3r}((x+y)/2)}\frac{h(z)}{|y-z|^{n-2}}dz\right| = I_1+I_2+I_3. \qquad (68)
\end{aligned}$$

We need to estimate I_1, I_2 and I_3 separately.

We begin to estimate

$$\begin{aligned}
I_2 \le \frac{M}{(n-2)\omega_n}\int_{B_{3r}((x+y)/2)}\frac{1}{|x-z|^{n-2}}dz \le \\
\le \frac{M}{(n-2)\omega_n}\int_{B_{4r}(x)}\frac{1}{|x-z|^{n-2}}dz, \qquad (69)
\end{aligned}$$

since the integrand is positive and $B_{3r}((x+y)/2) \subset B_{4r}(x)$. Changing to polar coordinates in (69) gives

$$I_2 \le \frac{8M}{(n-2)}r^2.$$

Interchanging the roles of x and y we may estimate I_3 in exactly the same way as we estimated I_2:

$$I_3 \le \frac{8M}{(n-2)}r^2.$$

It remains to estimate I_1. In order to do that we begin with a simple geometric estimate. By a translation of the coordinate system we may assume that $(x+y)/2 = 0$. If $z \in \mathbb{R}^n \setminus B_{3r}((x+y)/2) = \mathbb{R}^n \setminus B_r(0)$ then, for $t \in [0,1]$,

$$|tx+(1-t)y-z| \ge |z| - |tx+(1-t)y| \ge |z| - r > \frac{|z|}{2}, \qquad (70)$$

since $|z| \ge 3r$ and $|tx+(1-t)y| = |(2t-1)x| < r$ if $x+y=0$ and $|x-y| = r$.

Using the fundamental theorem of calculus we can also estimate

$$\left|\frac{1}{|x-z|^{n-2}}-\frac{1}{|y-z|^{n-2}}\right| = \left|\int_0^1 \frac{d}{dt}\frac{1}{|tx+(1-t)y-z|^{n-2}}dt\right| =$$

$$= (n-2)\left|\int_0^1 \frac{(x-y)\cdot(tx+(1-t)y-z)}{|tx+(1-t)y-z|^n}dt\right| \leq \tag{71}$$
$$\leq (n-2)|x-y|\int_0^1 \frac{1}{|tx+(1-t)y-z|^{n-1}}dt \leq$$
$$\leq \frac{2^{n-1}(n-2)|x-y|}{|z|^{n-1}},$$

where we used (70) in the last inequality.

Using (71) we may estimate

$$I_1 = \frac{1}{(n-2)\omega_n}\left|\int_{\mathbb{R}^n\setminus B_{3r}(0)}\left(\frac{h(z)}{|x-z|^{n-2}} - \frac{h(z)}{|y-z|^{n-2}}\right)dz\right| \leq$$
$$\leq \frac{M}{(n-2)\omega_n}\left|\int_{\mathrm{spt}(h)\setminus B_{3r}(0)}\left(\frac{1}{|x-z|^{n-2}} - \frac{1}{|y-z|^{n-2}}\right)dz\right| \leq$$
$$\leq \frac{2^{n-1}M|x-y|}{\omega_n}\int_{\mathrm{spt}(h)\setminus B_{3r}(0)} \frac{1}{|z|^{n-1}}dz. \tag{72}$$

The last integral in (72) can be estimated by noticing that[28]

$$\int_{\mathrm{spt}(h)\setminus B_{3r}(0)} \frac{1}{|z|^{n-1}}dz \leq \int_{B_{\mathrm{diam}(D)}} \frac{1}{|z|^{n-1}}dz =$$
$$= \omega_n \int_0^{\mathrm{diam}(D)} ds = \omega_n \mathrm{diam}(D),$$

where we changed to polar coordinates in the final step. Using this in (72) we can conclude that

$$I_1 \leq 2^{n-1}M|x-y|\mathrm{diam}(D).$$

Inserting the estimates of I_1, I_2 and I_3 in (68) we can conclude that

$$|u(x)-u(y)| \leq \frac{16M}{(n-2)}|x-y|^2 + 2^{n-1}M|x-y|\mathrm{diam}(D) \leq$$
$$\leq \left(\frac{16}{(n-2)} + 2^{n-1}\right)\mathrm{diam}(D)M|x-y|. \tag{73}$$

Noticing that the quantity in the brackets in (73) only depend on the dimension we may call that quantity c_n. This proves (66) □

[28]The integral increases if spt(h) is centered at the origin where $1/|z|^{n-1}$ is large. The integral therefore achieves its maximum if spt$(h) = B_{\mathrm{diam}(D)(0)}$.

Lemma 5.2. *Assume that $h(x)$ is a bounded, $|h(x)| \leq M$, and integrable function in $B_3(0)$ and that*

$$\begin{array}{ll} \Delta u = h(x) & \text{in } B_3(0) \\ \sup_{B_3(0)} u(x) \leq N. & \end{array}$$

Then

$$|u(x) - u(y)| \leq C_n\,(M+N)\,|x-y| \qquad \text{for all } x, y \in B_2(0).$$

Proof: If we define

$$v(x) = -\frac{1}{(n-2)\omega_n}\int_{B_3(0)} \frac{h(z)}{|x-z|^{n-2}}dz,$$

then, by Lemma 5.1,

$$|v(x) - v(y)| \leq C_n M|x-y|. \tag{74}$$

And, since $v(x)$ is defined by means of a convolution by the Newtonian kernel, $\Delta v(x) = h(x)$.

Also the function $w(x) = u(x) - u(y)$ satisfies

$$\Delta w(x) = \Delta u(x) - \Delta v(x) = h(x) - h(x) = 0$$

and

$$\sup_{B_3(x)} |w(x)| \leq \sup_{B_3(x)} |u(x)| + \sup_{B_3(x)} |v(x)| \leq N + \frac{9M}{2(n-2)},$$

where we used Lemma (5.1) in the last inequality.

Since $w(x)$ is harmonic we may use the following estimate[29] of the derivatives of $w(x)$

$$|\nabla w(x)| \leq C\|w\|_{L^1(B_1(x))} \leq C_n(N+M) \qquad \text{for any } x \in B_2(0). \tag{75}$$

From (75) and the mean value property for the derivative we can conclude that for any $x, y \in B_2(0)$ there exists a ξ between x and y such that.

$$|w(x) - w(y)| \leq |\nabla w(\xi)||x-y| \leq C_n(N+M)|x-y| \tag{76}$$

Finally we may use (74), (76) and the triangle inequality to conclude that for any $x, y \in B_2(0)$

$$|u(x) - u(y)| = |(w(x) - w(y)) + (v(x) - v(y))| \leq$$

$$\leq |w(x) - w(y)| + |v(x) - v(y)| \leq C_n\,(M+N)\,|x-y|,$$

where C_n may be different from the constant C_n in (76). □

[29] See for instance Theorem 7 in section 2.2 in Evans

Corollary 5.1. *Let $h^k(x)$ be a sequence of uniformly bounded, $|h^k(x)| \leq M$, and integrable functions and $u^k(x)$ be a sequence of functions that satisfies*

$$\begin{array}{ll} \Delta u^k = h^k(x) & \text{in } B_3(0) \\ \sup_{B_3(0)} u^k(x) \leq N. & \end{array}$$

Then there exists a function u^0 and a subsequence u^{k_j} such that $u^{k_j} \to u^0$ uniformly in $B_2(0)$

Proof: The sequence u^k is equicontinuous by Lemma 5.2. By the Arzela-Ascoli Theorem we may extract a uniformly converging sub-sequence. □

Corollary 5.2. *Let $u(x)$ be a solution to the normalized obstacle problem in D. Then the set $\Omega = \{x \in D;\ u(x) > 0\}$ is open.*

Proof: This follows from the continuity of the solution to $\Delta u(x) = \chi_{\{u>0\}}$. □

Lemma 5.3. [Comparison Principle.] *Let $f(x)$ and $g(x)$ be bounded functions in a bounded domain D. Furthermore assume that $\Delta u(x) = f(x)$ and $\Delta v(x) = g(x)$ in D.*[30] *Then if $f(x) \geq g(x)$ in D and $u(x) \leq v(x)$ on ∂D it follows that*

$$u(x) \leq v(x) \qquad \text{in } D$$

Proof:[31] It is enough to show that $w(x) = u(x) - v(x) \leq 0$ in D. That is we need to show that any function $w(x)$ that satisfies

$$\begin{array}{ll} \Delta w(x) = f(x) - g(x) \geq 0 & \text{in } D \\ w(x) = u(x) - v(x) \leq 0 & \text{on } \partial D \end{array} \tag{77}$$

will be non-positive.

Notice that w is the minimizer of

$$\int_D \left(\frac{1}{2} |\nabla w(x)|^2 + w(x)(f(x) - g(x)) \right) dx \tag{78}$$

in $K = \{w \in W^{1,2}(D);\ w = u(x) - v(x) \text{ on } \partial D\}$ since the Euler-Lagrange equations of (78) is $\Delta w(x) = f(x) - g(x)$ and the solution is unique.

By the function $\tilde{w}(x) = \min(w(x), 0) \in K$ and clearly

$$\int_D \left(\frac{1}{2} |\nabla \tilde{w}(x)|^2 + \tilde{w}(x)(f(x) - g(x)) \right) dx = \tag{79}$$

$$= \int_{D \cap \{w \leq 0\}} \left(\frac{1}{2} |\nabla \tilde{w}(x)|^2 + \tilde{w}(x)(f(x) - g(x)) \right) dx = \tag{80}$$

$$= \int_{D \cap \{w \leq 0\}} \left(\frac{1}{2} |\nabla w(x)|^2 + w(x)(f(x) - g(x)) \right) dx \leq \tag{81}$$

[30] Assume for instance that $u, v \in W^{2,2}(D)$ in order to make sense of these equations.
[31] The proof is more or less the same as in a previous exercise.

$$\leq \int_D \left(\frac{1}{2} |\nabla w(x)|^2 + w(x)(f(x) - g(x)) \right) dx, \tag{82}$$

with equality only if $w(x) \leq 0$ in D. But since w is a minimizer we must have equality in (79)-(82). This finishes the proof. □

Theorem 5.1. *Let $u(x)$ be a solution to the normalized obstacle problem in the domain D. Assume furthermore that for some $s > 0$*

$$x^0 \in \Gamma \cap \{x \in D;\ dist(x, \partial D) > s\}.$$

Then there exists a constant C, depending on the dimension n and on N, such that

$$\sup_{x \in B_r(x^0)} u(x) \leq Cr^2 \qquad \text{for every } r \leq \frac{s}{2},$$

where the constant C depend only on the dimension.

Remark: *Notice that the constant C does not depend on the solution.*
Proof: We will prove the Theorem in several steps.

Step 1: *Reduction to the statement that it is enough to prove that: If $u(x)$ is a solution to the normalized obstacle problem in $B_2(0)$ such that $u(0) = 0$ then $\sup_{B_1(0)} u \leq C$ for some C depending only on the dimension.*

Proof of step 1: Let $u(x)$ and x^0 is as in the Theorem. Then the function

$$u_r(x) = \frac{u(rx + x^0)}{r^2}$$

will satisfy $u_r \in W^{2,2}(B_2(0))$ and

$$\Delta u_r(x) = \Delta \left(\frac{u(rx + x^0)}{r^2} \right) = \Delta u(y) \lfloor_{y = rx + x^0} =$$

$$= \left\{ \begin{array}{l} 1 \text{ if } u(rx + x^0) > 0 \Rightarrow u_r(x) > 0 \\ 0 \text{ if } u(rx + x^0) = 0 \Rightarrow u_r(x) = 0, \end{array} \right\} = \chi_{\{u_r > 0\}}$$

in the set $x \in \{x;\ rx + x^0 \in D\}$. This is a convoluted way of trying to indicate how the chain rule implies that

$$\Delta u_r(x) = \chi_{\{u_r(x) > 0\}} \qquad \text{in } \{x;\ rx + x^0 \in D\}.$$

Notice that if $B_s(x^0) \subset D$ then $B_2(0) \subset \{x;\ rx + x^0 \in D\}$. Thus u_r solves the normalized obstacle problem in $B_2(0)$ and $u_r(0)$.

Thus if any solution $v(x)$ to the normalized obstacle problem in $B_2(0)$ that satisfies $v(0) = 0$ satisfies $\sup_{B_1(0)} v(x) \leq C$ then this applies to u_r. We may conclude that

$$\frac{u(rx + x^0)}{r^2} = u_r(x) \leq C \qquad \text{for every } x \in B_1(0).$$

But this implies that

$$\sup_{x\in B_r(x^0)} u(x) \leq C.$$

Step 1 is therefore proved.

Step 2: *If $u(x)$ is a solution to the normalized obstacle problem in $B_2(0)$ and $u(0)=0$ then ther exists a constant c_n such that if $y \in \partial B_1(0)$ then*

$$u(x) \geq c_n u(y) - \frac{1}{2n} \qquad \text{for all } x \in B_{1/2}(0) \cap \partial B_1(0).$$

Proof of Step 2: Since $\Delta u(x) \leq 1$ it follows from the comparison principle that $u(x) \leq v(x)$ where $v(x)$ is defined by

$$\begin{array}{ll} \Delta v(x) = 0 & \text{in } B_1(y) \\ v(x) = u(x) & \text{on } \partial B_1(y). \end{array} \tag{83}$$

If we define $w(x) = v(x) - \frac{1}{2n} + \frac{1}{2n}|x-y|^2$ then

$$\begin{array}{ll} \Delta w(x) = 1 \geq \Delta u(x) & \text{in } B_1(y) \\ w(x) = u(x) & \text{on } \partial B_1(y). \end{array}$$

We may conclude that

$$v(x) - \frac{1}{2n} \leq w(x) \leq u(x) \leq v(x) \qquad \text{in } B_1(y). \tag{84}$$

Since $v(x) \geq u(x) \geq 0$ and $v(y) \geq u(y)$ we may conclude from the Harnack inequality that, for some constant C_n only depending on the dimension,

$$v(y) \leq \sup_{B_{1/2}} v(x) \leq C_n \inf_{B_{1/2}(y)} v(x) \Rightarrow \inf_{B_{1/2}(y)} v(x) \geq \frac{v(y)}{C_n} \geq \frac{u(y)}{C_n}, \tag{85}$$

where we also used that $v(y) \geq u(y)$ in the last inequality.

But from (84) and (85) we can conclude that

$$\frac{u(y)}{C_n} \leq \inf_{B_{1/2}(y)} v(x) \leq \inf_{B_{1/2}(y)} u(x) + \frac{1}{2n} \Rightarrow \frac{u(y)}{C_n} - \frac{1}{2n} \leq \inf_{B_{1/2}(y)} u(x).$$

This proves step 2 with $c_n = \frac{1}{C_n}$.

Step 3: *Assume that $u(x)$ is a solution to a normalized obstacle problem in $B_2(0)$ such that $u(0)=0$ then $\sup_{B_1(0)} u(x) \leq C_n$ for some universal constant C_n depending only on the dimension.*

Proof of Step 3. If we let $h(x)$ be the function defined by

$$\begin{array}{ll} \Delta v(x) = 0 & \text{in } B_1(0) \\ v(x) = u(x) & \text{on } \partial B_1(0). \end{array} \tag{86}$$

Then we may argue as in step 2 to conclude that that $v(x) - \frac{1}{2n} \leq u(x) \leq v(x)$. Since $u(0) = 0$ we can conclude that

$$v(0) \leq \frac{1}{2n}. \tag{87}$$

By the mean-value property for harmonic functions we can conclude from (87) that

$$\frac{1}{2n} \geq \frac{1}{\omega_n} \int_{\partial B_1(0)} v(x)dx = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x)dx, \tag{88}$$

since $v(x) = u(x)$ on $\partial B_1(0)$.

If $u(y) = \sup_{x \in \partial B_1(0)} u(x)$ then we may estimate the right side in (88) according to

$$\frac{1}{2n} \geq \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x)dx \geq \frac{1}{\omega_n} \int_{\partial B_1(0) \cap B_{1/2}(y)} u(x)dx \geq \tag{89}$$

$$\geq \frac{1}{\omega_n} \int_{\partial B_1(0) \cap B_{1/2}(y)} \inf_{z \in B_{1/2}(y)} u(z)dx \geq \frac{K}{\omega_n} \left(c_n u(y) - \frac{1}{2n} \right),$$

where $K = \int_{\partial B_1(0) \cap B_{1/2}(y)} dA$ and we used that $u \geq 0$ in the second inequality and Step 2 as well as the fact that $B_{1/2}(y) \cap \partial B_1(0)$ consists of a fixed proportion of $\partial B_1(0)$ in the last inequality.

Rearranging the terms in (89) we arrive at

$$u(y) \leq \frac{K+1}{2Kc_n},$$

where the right side depend only on the dimension. This finishes the proof. □

Corollary 5.3. *If u is a solution to the normalized obstacle problem in a domain D. Then $|\nabla u(x)| = 0$ for any point $x \in \Gamma$.*

Proof: Let $x^0 \in \Gamma \cap D$. We need to show that

$$\lim_{x \to x^0} \frac{u(x) - u(x^0)}{|x - x^0|} = 0.$$

But if we use the notation $r = r(x) = |x - x^0|$ then it directly follows from Theorem 5.1 and the assumption $x^0 \in \Gamma$ (which implies that $u(x^0) = 0$ since u is continuous by Lemma 5.2) that

$$\left| \frac{u(x) - u(x^0)}{|x - x^0|} \right| \leq \frac{Cr^2}{r} = Cr \to 0 \text{ as } r \to 0.$$

The Corollary follows. □

Theorem 5.2. *Let $u(x)$ be a solution to the normalized obstacle problem in a domain D. Then there exists a constant C_n depending only on the dimension such that if $u(y)=0$ and $B_{s/4}(y)\subset D$ then*

$$|D^2u(x)|\le C_n \qquad \text{for every } x\in B_{s/8}(y)\cap\{u>0\}.$$

Furthermore, $u(x)$ is analytic in $\Omega=\{u(x)>0\}$.

Proof: Let $y\in D$ be any point such that $u(y)=0$. Also let $s=\operatorname{dist}(y,\partial D)$ so that $B_s(y)\subset D$ and $z\in B_{s/8}(y)\cap\{u>0\}$.

Next we consider the largest ball $B_r(z)\subset\{u>0\}$ and pick any point $q\in\partial B_r(z)\cap\partial\{u>0\}$. Notice that since $z\in B_{s/8}(y)$ and $u(y)=0$ it follows that $r\le s/8$.

We also claim that $B_{4r}(q)\subset D$. By the triangle inequality $|y-q|\le|y-z|+|z-q|<s/8+r<s/2$ which implies that $B_{4r}(q)\subset B_{s/2}(q)\subset B_s(y)\subset D$.

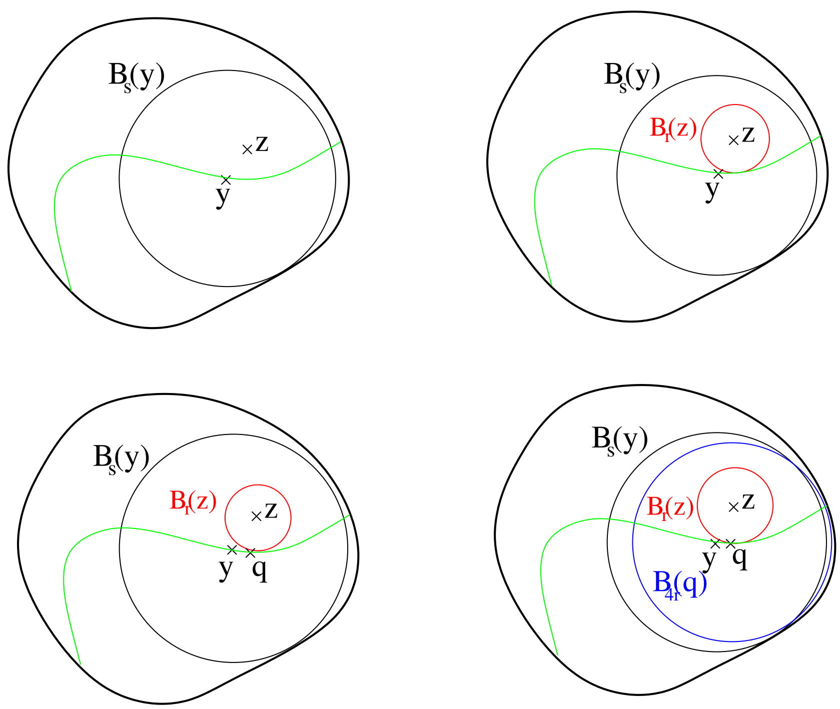


Figure 10: *The above figure (not drawn to scale) tries to indicate how $B_r(z)$ and $B_{4r}(q)$ is choosen. We have the point y such that $B_s(y)\subset D$ shown in the first figure. The ball $B_r(z)$ (in red) is then choosen to be the largest ball contained in $\Omega=\{u>0\}$. That means that $\partial B_r(z)$ touches the free boundary (the green curve) in some point q as shown in the third picture. We then choose the ball $B_{4r}(q)$ (in blue) as in the last picture. The purpose of this construction*

is that since $B_{4r}(q) \subset D$ we know, Theorem 5.1, that $u(x)$ is uniformly bounded in $B_r(z) \subset B_{4r}(q)$.

By Theorem 5.1 it follows that

$$\sup_{x \in B_{2r}(q)} u(x) \leq 4Cr^2.$$

This in particular implies that

$$\sup_{x \in B_r(z)} u(x) \leq \sup_{x \in B_{2r}(q)} u(x) \leq 4Cr^2$$

since $B_r(z) \subset B_{2r}(q)$.

This implies that

$$\left.\begin{array}{l} \Delta u(x) = 1 \text{ in } B_r(z) \\ u(x) \leq 4Cr^2 \text{ in } B_r(z) \end{array}\right\} \Rightarrow \left\{ \begin{array}{ll} \Delta\left(u(x) - \frac{1}{2n}|x-z|^2\right) = 0 & \text{in } B_r(z) \\ \left|u(x) - \frac{1}{2n}|x-z|^2\right| \leq \left(4C + \frac{1}{2n}\right) r^2 & \text{in } B_r(z). \end{array}\right.$$

Using standard estimates on derivatives for harmonic functions[32] we can conclude that, at the point $x = z$,

$$\left| D^2 u(x) - \frac{1}{2n}|x-z|^2 \right| \leq C_2 \left(4C + \frac{1}{2n} \right).$$

But this clearly implies that

$$|D^2 u(z)| \leq C_n,$$

where C_n is a constant that only depend on the dimension.

That u follows from the fact that $u - \frac{1}{2n}|x-z|^2$ is harmonic in a small neighborhood around z and that harmonic functions are analytic. □

Exercises:

1. * Let $u(x)$ be a minimizer of the normalized obstacle problem in $B_1(0) \subset \mathbb{R}^3$ with constant boundary values $u(x) = t$ on $\partial B_1(0)$. Calculate $u(x)$ and show that the free boundary Γ is given by a sphere of radius s. Determine the relation between t and s.

   HINT: *If we write $u(x) = u(r)$ where $r = |x|$ then we get a one dimensional problem with the following conditions $u(1) = t$, $u(s) = |\nabla u(s)| = 0$. Since $u(r) - \frac{1}{6}r^2$ is harmonic in $\{u(r) > 0\}$ we should be able to write $u(r) - \frac{1}{6}r^2 = \frac{c}{r} + d$ for two constants c and d. Since also s is unknown we have three unknown and three boundary conditions to satisfy.*

2. Verify all the calculations in the proof of Lemma 5.1.

[32] $|D^2 h(z)| \leq \frac{C_2}{r^{n+2}} \|h\|_{L^1(B_r(z))} \leq \frac{C_2}{r^2} \|h\|_{L^\infty(B_r(z))}$ see Evans Theorem 7 chapter 2.2.

3. ** Let $u(x)$ be a solution to the normalized obstacle problem in a connected domain D. Show that for any set $\mathcal{C}$ such that $\mathrm{dist}(\mathcal{C}, \partial D) \geq \delta > 0$ there exists a constant C such that if $u(x) \geq C$ for any point $x \in \mathcal{C}$ then $\Gamma \cap \mathcal{C} = \emptyset$.

4. ** Let $u(x)$ be a solution to the normalized obstacle problem in D. Show that if $x^0 \in \Gamma$ then $\sup_{x \in B_r(x^0)} u(x) \geq \frac{r^2}{2n}$.

   HINT: *Let y be a point arbitrarily close to x^0 such that $u(y) > 0$. Argue by contradiction and assume that $u(y) - \frac{1}{2n}|x-y|^2$ is strictly negative on $\partial B_r(y)$. What equation does $u(y) - \frac{1}{2n}|x-y|^2$ solve in $\Omega \cap B_r(y)$? What are the boundary values of $u(y) - \frac{1}{2n}|x-y|^2$ on $\partial(\Omega \cap B_r(y))$?*