Mid-term exam 2 for HF1013, Discrete Mathematics:TEN1, 5 credits, fall 2015

Solutions

1. If p and q are two different prime numbers, prove that there are always integers s, t such that

$$sp^n + tq^n = n \tag{1}$$

for every integer n.

Solution: As p, q are different prime numbers they must be relatively prime, but the same must also apply to p^n and q^n, hence there exists integers, s', t' such that

$$s' \cdot p^n + t' \cdot q^n = 1.$$

. But then, by simply putting $s = n \cdot s'$ and $t = n \cdot t'$ we obtain (1).

2. Use mathematical induction to show that 5 divides $n^5 - n$ whenever n is a nonnegative integer.

Solution: Introduce the predicate $A(n)$ for the statement that 5 divides $n^5 - n$. Now take the three steps in a proof by mathematical induction.

1. Check that $A(0)$ holds, that is check that $5|0^5 - 0$. But this clearly holds since it is the same thing as $5|0$ which is certainly true since 0 is divisible by any number.
2. Show that the implication $A(p) \Rightarrow A(p+1)$ is true for all $p \geq 0$. Assume that $A(p)$ holds, that is assume that there is a k such that $p^5 - p = 5 \cdot k$. We must now show that $A(p+1)$ that is there is a k' such that $(p+1)^5 - (p+1) = 5 \cdot k'$ for some k'. Now

$$(p+1)^5 - (p+1) = p^5 + 5p^4 + 10p^3 + 10p^2 + 5p + 1 - p - 1 = p^5 - p + 5 \cdot (p^4 + 2p^3 + 2p^2 + p).$$

We have yet not used the induction assumption, but we do that by replacing $p^5 - p$ with $5 \cdot k$ in the expression above. This leads to

$$(p+1)^5 - (p+1) = 5 \cdot k + 5 \cdot (p^4 + 2p^3 + 2p^2 + p) = 5 \cdot (k + p^4 + 2p^3 + 2p^2 + p) = 5 \cdot k'$$

where $k' = k + p^4 + 2p^3 + 2p^2 + p$. This is just another way of saying that $5|(p+1)^5 - (p+1)$ which is precisely the statement $A(p+1)$. This means that we have shown that the implication $A(p) \Rightarrow A(p+1)$ is true for each $p \geq 1$.
3. Step 1 gives that $A(0)$ is true, by step 2 this implies that $A(1)$ is true which by step 2 again gives that $A(2)$ is true and so on showing, by the principle of mathematical induction (relying on step 1 and 2), that for each $n \geq 0$ the statement $A(n)$ is true which completes the proof.