Recommended exercises section 0.1 (1.1) Do all [BB]-ones, further:

2b (1b), 2j (1d), 2q (2b), 5b (3b), 5d (3d), 7b (5b), 8 (6)

Solutions:

2b (1b): $4 \neq 2+2$ and $7 < \sqrt{50}$. This is false since $4 \neq 2+2$ is false. ($4 \neq 2+2$ or $7 < \sqrt{50}$ would have been true.)

2j (1d): $4 = 2+2 \leftrightarrow 7 < \sqrt{50}$. This is true since both $4 = 2+2$ and $7 < \sqrt{50}$ are true.

2q (2b): "If a and b are integers with $a-b>0$ and $b-a>0$, then $a=b$." This looks strange but in fact it is true. The statement is of the form $P \to Q$ where $P =$ "$a-b>0$ and $b-a>0$" and $Q =$ "$a=b$". But if we study P we see that it is equivalent to $a>b$ and $b<a$ and since a & b are integers this must be false. A strange proof of this might go like this: Since a & b are integers we must have

$a<b$ or $a=b$ or $b<a$

Since $a<b$ we cannot have $b<a$ so we cross that out and are left with $a<b$ or $a=b$. But since also $b<a$ we cannot have $a<b$ which leaves us with $a=b$. That is we see that the implication $P \to Q$ is true. That P itself is not true ever is not part of these considerations. A similar bizarre example can be seen by googling for "Bertrand Russel proof pope".

5b (3b): Negate "x is a real number and $x^2+1=0$". This statement is of the form "p and q" and so the negation must be of the form "$\neg p$ or $\neg q$" which reads

x is not a real number OR $x^2+1 \neq 0$

5d (3d): Negate "Every integer is divisible by a prime". This statement is of the form "$\forall x : \exists p :$ x is divisible by p" (x stands for integers and p for primes.) Negating this is done in two steps:

$\neg \forall x : \exists p :$ x is div. by p $\leftrightarrow \exists x : \neg \exists p :$ x is div. by p $\leftrightarrow \exists x : \forall p : \neg$(x is div by p)

which reads "there is an integer which is not divisible by any prime."

7b(5b): "For real x, 2^x is never negative" can be written

$\forall x : 2^x \geq 0$ (x stands for integers.)

8(6): Is it possible for both an implication and its converse to be false?

An implication is of the form $p \to q$, the converse is $q \to p$. Can both be false?

We can divide this discussion into two cases, either p is true or p is false. If p is true, then $q \to p$ must be true regardless of whether or not q is true or false (this is the nature of the implication). This means that not both $p \to q$ and $q \to p$ can be false. In the other case when p is false, then $p \to q$ must be true (again this is in the nature of an implication) so that not both $p \to q$ and $q \to p$ can be false. In both cases (p true or false) not both $p \to q$ and $q \to p$ can be false so the answer is NO.

(Remark: If "the nature of an implication" feels strange, just accept it for now.)

Recommended exercises section 0.2 (1.2) Do all [BB]-ones, further:

8(NA), 15(NA), 17(12), 30(23)

Solutions:

8(NA): Consider the statement A: "If n is an integer $\frac{n}{n+1}$ is not an integer."

(a) Is A true or false? Prove true or give counterexample.

A is false because there exists an integer n for which $\frac{n}{n+1}$ is an integer and this integer is $n=0$ for which $\frac{n}{n+1} = \frac{0}{0+1} = 0$. This is a counterexample. (If we would have considered only positive integers $1, 2, 3, \dots,$ then A would have been true – proof by contradiction.)

(b) A: "n is an integer $\rightarrow$ $\frac{n}{n+1}$ is not an integer". $(p \rightarrow q)$

Converse: $\frac{n}{n+1}$ is not an integer $\rightarrow$ n is an integer $(q \rightarrow p)$
(false since $n = 1/2$ gives a counterexample)

Contrapositive: $\frac{n}{n+1}$ is an integer $\rightarrow$ n is not an integer $(\neg q \rightarrow \neg p)$
(false since $n=0$ is a counterexample.)

Negation: n is an integer and $\frac{n}{n+1}$ is an integer $(p \wedge \neg q)$
(true for $n = 0$.)

15(NA): (a) Prove that "n is an even integer $\rightarrow$ n^2+3n is an even integer"

Proof: n even integer $\Leftrightarrow \exists k,\ n = 2k$, where k is an integer $\Rightarrow$
$n^2+3n = n(n+3) = 2k(2k+3)$ and this is clearly an even integer.
The proof is complete.

(b) The converse is n^2+3n is an even integer $\Rightarrow$ n is even.
Is it true? Either we can prove that it is true or we can find a counterexample. Try to prove that it is true, you will fail but it can be instructive to try. It is false because $n=3$ has $n^2+3n = 18$ which is even, but n is odd.

17(12): Provide a direct proof that n^2-n+5 is odd for all integers n.

Proof: Since $n^2-n = n(n-1)$ we see that it is a product of two consecutive numbers, one of them must be even so that $n(n-1) = 2k$ for some k. But then $n^2-n+5 = 2k+5 = 2(k+2)+1$ which clearly is an odd number. The proof is complete.

30(23): Suppose that a is a rational number and that b is an irrational number. Prove that $a+b$ is irrational.

Proof: We proceed by proof by contradiction and therefore we assume the opposite of what we want to prove, that is we assume that $a+b$ is rational. Then there exist integers p, q such that

$$a+b = \frac{p}{q}.$$

As a also was rational, we have $a = \frac{m}{n}$ for integers m, n. But this then yields

$$\frac{m}{n} + b = \frac{p}{q} \iff b = \frac{p}{q} - \frac{m}{n} = \frac{pn - mq}{qn} = \frac{\text{integer}}{\text{integer}}$$

which means that b is rational, but b was irrational. We conclude that the assumption must be false, that is it must be false that $a+b$ is rational, that is $a+b$ is irrational which is what we wanted to prove.

Remark: This is a standard proof by contradiction — very important to grasp. If you have trouble understanding it (which many will have) study the section on proof by contradiction and in particular problems 6, 7, 8 page 14.