Recomended review exercises Chapter 0 (1):

2(2), 3(3), 9(6), 15(NA)

Solutions:

2: Write down the negation of each of the following statements in clear and concise English:

a) Either x is not a real number or $x > 5$ ($\sim(x \text{ real}) \vee x > 5$)

x is real and $x \leq 5$.

b) There exists a real number x such that $n > x$ for every integer n. ($\exists x$ real : $\forall n$: n integer $\rightarrow n > x$)

$\forall x$ real : $\exists n$: n integer $\wedge$ $n \leq x$

For all real x we can find an integer n that has $n \leq x$.

c) For all x, y, z positive integers we have $x^3 + y^3 \neq z^3$

$(\forall x : \forall y : \forall y : x^3 + y^3 \neq z^3)$ $\exists x \exists y \exists z : x^3 + y^3 = z^3$

There exist positive integers x, y, z such that $x^3 + y^3 = z^3$.

3: Write down the converse, the contrapositive and the negation of each of the following implications. ($p \rightarrow q$, $q \rightarrow p$, $\sim q \rightarrow \sim p$, $p \wedge \sim q$)

a) If a & b are integers, then ab is an integer

$q \rightarrow p$: If ab is an integer, then a & b are integers. (false)

$\sim q \rightarrow \sim p$: If ab is not an integer, then ONE OF a & b is not an integer (true)

$p \wedge \sim q$: There is integers a & b for which ab is not an integer (false)

b) If x is an even integer then x^2 is an even integer

$q \rightarrow p$: If x^2 is an even integer, then x is an even integer.

$\sim q \rightarrow \sim p$: Ix x^2 is not an even int, then x is not an even int.

$p \wedge \sim q$: There is an even integer x with x^2 not being an even integer

9(6): Let n be an integer. Prove that n^3 is odd if and only if n is odd.

We need to prove n^3 odd $\Leftrightarrow$ n odd that is n^3 odd $\rightarrow$ n odd and n odd $\rightarrow$ n^3 odd.

n odd $\rightarrow$ n^3 odd: Assume n is odd. Then $n = 2x+1$ for some integer x. Then $n^3 = (2x+1)^3 = 8x^3 + 12x^2 + 6x + 1 = 2(4x^3 + 6x^2 + 3x) + 1$ which is an odd number. We have shown n odd $\rightarrow$ n^3 odd.

To prove n^3 odd $\rightarrow$ n odd we show the contrapositive (which is the same thing) which reads n not odd $\rightarrow$ n^3 not odd. Since for all integers an integer is not odd if and only if it is even this is the same as n even $\rightarrow$ n^3 even, which is what we want to prove. So assume n is even, then, there exists an integer x such that $n = 2x$. This gives $n^3 = (2x) = 8x^3 = 2 \cdot 4x^3$ which obviously is even. We have therefore shown n even $\rightarrow$ n^3 even which is the same thing as n^3 odd $\rightarrow$ n odd which completes the proof.

15 (NA): Prove that $\sqrt{3}$ is not a rational number

Proof: Assume that $\sqrt{3}$ is a rational number, then there are integers p, q with $\sqrt{3} = \frac{p}{q}$, $q \neq 0$ and p, q have no common factors. But then we have $3 = \frac{p^2}{q^2} \Leftrightarrow p^2 = 3q^2$ which shows that p must have a factor of 3 in it (this is true since 3 is not a square), this means that we can write $p = 3 \cdot k_1$ for some integer k_1. But then we get

$$(3 \cdot k_1)^2 = 3q^2 \Leftrightarrow 9k_1^2 = 3q^2 \Leftrightarrow 3k_1^2 = q^2$$

and by the same reasoning we get that also q has a factor of 3 in it, say $q = 3 \cdot k_2$. But then both p and q have a common factor 3 ($p = 3k_1$ and $q = 3k_2$) which contradicts that we can choose p & q with no common factors. Hence the assumption that $\sqrt{3}$ is a rational number must be false. The proof is complete.

Can you generalize this?