Recommended review exercises Chapter 1 (1)
1 (8), 4 (10), 6a (12), 8 (14)

Solutions:

1(8): Construct a truth table for the compound statement

$$[p \wedge (q \to (\neg r))] \to [(\neg q) \vee r]$$

The truth table for the implication $a \to b$

a	b	$a \to b$
T	T	T
T	F	F
F	T	T
F	F	T

will be used more than once. The atomic statements are p, q, r so we build the truthtable in several steps

p	q	r	$q \to (\neg r)$	$p \wedge (q \to (\neg r))$	$(\neg q) \vee r$	$[p \wedge (q \to (\neg r))] \to [(\neg q) \vee r]$
T	T	T	F	F	T	T
T	T	F	T	T	F	F
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	F	F	T	T
F	T	F	T	F	F	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T
			1.	2.	3.	4.

In the first step we establish column 4, then q has the role of a and $(\neg r)$ has the role of b. This means that $q \to (\neg r)$ is false precisely when q (above a) is true and $(\neg r)$ is false which is precisely when q is true and r is true. $q \to (\neg r)$ is true everywhere else. This results in column 4 above (marked 1.) In the second step we establish the next column (marked with 2). It is easier just to combine the truth values in the column for p with the column for $q \to (\neg r)$. The next column (marked 3) is also easy, just combine truth values for q & r with disjunction.) Finally we create column 4 which finalizes the solution. To do so we let column 2 take the role of a and column 3 take the role of b and mark the entries of column 4 as true precisely when either the truth value of the entry in column 2 (corresponding to a) is false or when both entries in both columns are true. This happens on all but the second row. This is the spirit of the implication: it is false only when the premise is true and the conclusion is false.

4(10): Two compound statements A and B have the property that $A \to B \Leftrightarrow B \to A$. what can you conclude about A and B?

We can write this as $(A \to B) \leftrightarrow (B \to A)$ and make a truth table

A	B	$A \to B$	$B \to A$	$(A \to B) \leftrightarrow (B \to A)$	$A \leftrightarrow B$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

The columns for $A \to B$ and $B \to A$ and so on are built up as before. we clearly see that if $(A \to B) \leftrightarrow (B \to A)$ then A & B always have the same truth value, that is $A \leftrightarrow B$ is true.

6a(12): Establish the logical equivalence of each of the following pair of statements: $[(p \to q) \to r]$ and $[(p \vee r) \wedge (\neg(q \wedge (\neg r)))]$

p	q	r	$p \to q$	$(p \to q) \to r$	$p \vee r$	$q \wedge (\neg r)$	$\neg(q \wedge (\neg r))$	$(p \vee r) \wedge (\neg(q \wedge (\neg r)))$
T	T	T	T	T	T	F	T	T
T	T	F	T	F	T	T	F	F
T	F	T	F	T	T	F	T	T
T	F	F	F	T	T	F	T	T
F	T	T	T	T	T	F	T	T
F	T	F	T	F	F	T	F	F
F	F	T	T	T	T	F	T	T
F	F	F	T	F	F	F	T	F
				⊛				⊙

We build the table incrementally (step by step) and compare the columns for the two statements that we wish to show are equivalent. They are marked ⊛ and ⊙ and as they are identical the statements are equivalent which is a complete proof.

8(14): Determine whether each of the following arguments is valid:

(a) $p \to q$
$\neg p$
$\neg q$

This is not a valid argument. To show that it is not valid we must find an assignment of truth values to p and q that satisfy the premises, $p \to q$ and $\neg p$, but not the conclusion $\neg q$. There is only one such assignment and it is

$p =$ false $\quad q =$ true.

(b) To study the next argument we label the premises.

1. $\neg((\neg p) \wedge q)$
2. $\neg(p \wedge r)$
3. $r \vee s$

$q \to s$

To show an implication we assume that q is true and try to show that s follows, then $q \to s$ is shown to be true:

4. q

Then we use various rules of inference, DeMorgan's laws on 1 & 2 gives

5. $\neg(\neg p) \vee \neg q$ (1, DM)
7. $\neg p \vee \neg r$ (2, DM)

Then we simply refer to numbers above and more inference rules, like this:

8. $\neg(\neg p)$ (4, 5, Disjunctive Syllogism.)
9. p (8, Double Negation)
10. $\neg r$ (7, 9, Disjunctive Syllogism.)
11. s (10, 3, Disjunctive Syllogism.)

We conclude that the assumption of q leads to s which means that $q \to s$ must be true.

12. $q \to s$ (4 - 11, Proof by assumption.)

The argument is valid.

8.(14):

(c)

An implication

a	b	$a \to b$
T	T	T
T	F	F
F	T	T
F	F	T

1. $p \vee (\neg q)$
2. $(t \vee s) \longrightarrow (p \vee r)$
3. $(\neg r) \vee (t \vee s)$
4. $p \Leftrightarrow (t \vee s)$

$(p \wedge r) \to (q \wedge r)$

At first one tries to prove the conclusion from the premises but it always fail, then the alternate strategy is to prove that the argument is not correct. To do this we must find an assignment of truth values to p, q, r, s and t so that 1,2,3,4 hold but $(p \wedge r) \to (q \wedge r)$ is false. The only way an implication is false is when the premise is true but the conclusion is false, for $(p \wedge r) \to (q \wedge r)$ this means we would have $p \wedge r$ and $\neg(q \wedge r) \Leftrightarrow (\neg q) \vee (\neg r)$ (De Morgan).

Can we find an assignment of truth values of p, q, r, s, t such that $p \wedge r$ and $(\neg q) \vee (\neg r)$ and also 1,2,3,4 hold? Obviously p = true and r = true. p = true makes 1 true by absorption. Similarly 2 is true since $p \vee r$ is true. (If a conclusion in an implication is true, then the implication becomes true — see the truth table above for b = true.) That r = true means we must have q = false (to have $(\neg q) \vee (\neg r)$ true), then in order for 3 to hold we must have $t \vee s$ true. Setting t = true makes both 3 and 4 true.

Answer: p = true, q = false, r = true, s = any value, t = true is an assignment of truth values which makes 1, 2, 3, 4 hold but not $(p \wedge r) \to (q \wedge r)$. The argument is not valid.