Recommended review exercises, Chapter 10

3(3), 4(4), 6(6), 12(8), 22(15)

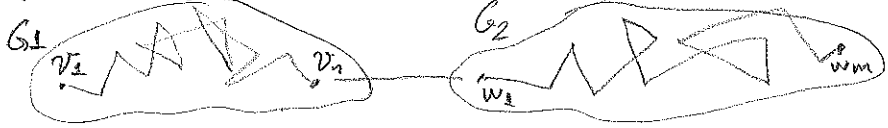
Solutions:

3(3): Suppose G_1 and G_2 are graphs with no vertices in common and assume that each graph possesses an Eulerian trail. (Each edge) Show that it is possible to select vertices v and w of G_1 and G_2 respectively such that if v and w are joined by a new edge the resulting graph will possess an Eulerian trail.

Proof: Denote the Eulerian trails of G_1 and G_2 by $v_1 v_2 \dots v_n$ and $w_1 w_2 \dots w_m$ respectively. Choose $v = v_n$, that is the last node in the Eulerian trail in G_1, and choose $w = w_1$, that is the first node in the Eulerian trail in G_2. When the edge e from v to w is added, it is clear that

$$v_1 v_2 \dots v_n w_1 w_2 \dots w_m$$

is the required Eulerian trail of the resulting graph



4(4): True or false? Explain your answers in each case.

(a) Every trail is a path. (trail: walk, distinct edges; path: walk, distinct vertices, trail $\Rightarrow$ path?)
<u>NO</u>, because it is possible to return to a previously visited vertex but not reusing edges.

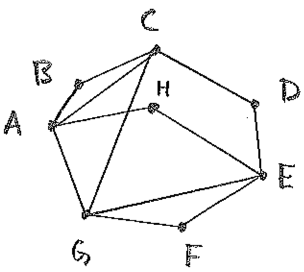
(b) Every open trail a path. (open: not beginning/ending at the same vertex)
<u>NO</u>, because the previous explanation holds here too.

(c) If there is an open trail from vertex v to vertex w, then there is a path from v to w.
<u>YES</u>, we can just drop any circuits contained in the trail from v to w resulting in a sequence of non-recurring vertices – a path.

(d) Every path is an open trail. <u>YES</u>, distinct vertices implies distinct edges.

(e) If there is a path from vertex v to vertex w, then there is an open trail from v to w. <u>YES</u>, In fact the path itself is the open trail, again since distinct vertices (requirement for path) implies distinct edges (requirement for open trail).

6(6): Is the graph Hamiltonian? Is it Eulerian? Explain your answers.



The graph is Eulerian since the degree of each vertex is even. An Euler-cycle would for example be

GFEDCBAGEHACG

The graph is NOT Hamiltonian since if it would contain a Hamiltonian cycle, E would be in that cycle. However, then the cycle would have one of the following vertices occurring in that order: HED, GED, FED, HEG, HEF, GEF. We shall see that each of these combinations make it impossible that all vertices are included in the Hamiltonian cycle, since one central requirement is that no vertex occurs twice.

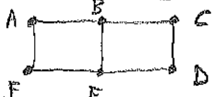
- HED: In this case, the vertex F cannot be reached since this would require us to traverse the edge from F to E but that is forbidden since E is already visited.
- GED: The same argument as for HED excludes vertex F.
- FED: Here H is excluded.
- HEG: Here again F is excluded.
- HEF: Here D is excluded.
- GEF: Here H is excluded. (And D.)

In every possible case there is no Hamiltonian cycle, hence the graph is not Hamiltonian.

12(8) True or false? Explain your answers in each case.

(a) In a Hamiltonian graph, every edge belongs to a cycle.

False, consider this Hamiltonian graph:



The edge from B to E cannot be in a Hamiltonian cycle since it would cut the graph in two.

(b) In a Hamiltonian graph, every edge belongs to a cycle

True, for an arbitrary edge, e, either e is in a Hamiltonian cycle or it connects two nodes that are in a Hamiltonian cycle. In the latter case e is in the cycle formed by going across e and then following the edges of the Hamiltonian cycle back to the beginning of e.

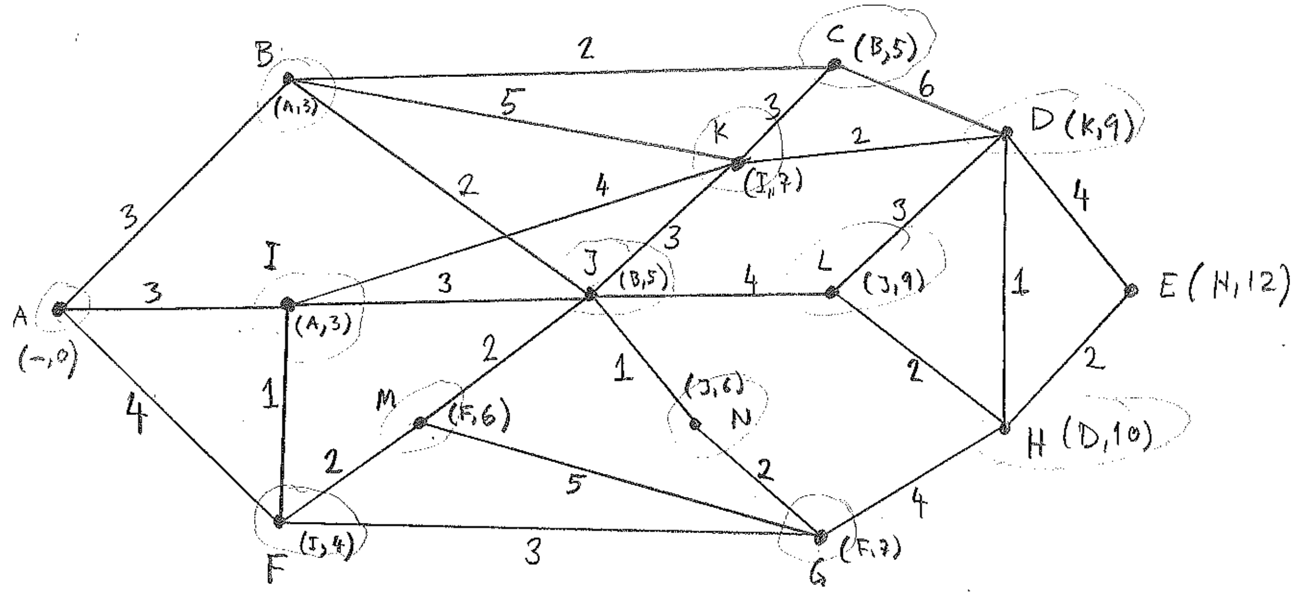
(c) Every Eulerian graph contains a subgraph that is Hamiltonian

True, just choose any cycle with non-recurring vertices.

(d) Every Hamiltonian graph contains a subgraph that is Eulerian

True, just take the Hamiltonian cycle with its vertices and edges and remove all edges that were not traversed.

22(15): Apply the first form of Dijkstra's algorithm to the following graph, showing the shortest distance from A to every other vertex. Exhibit an order in which a shortest path from A to E might be realized.



A: B: 0+3 = 3 I: 0+3 = 3 F: 0+4 = 4 Minimum: 3, for B & I

ABI: C: 3+2 = 5 K: 4+3 = 7 J: 3+2 = 5 F: 3+1 = 4

Minimum: 4, for F

ABFI: C: 3+2 = 5 K: 4+3 = 7 J: 3+2 = 5 M: 2+4 = 6 G: 3+4 = 7

Minimum: 5 for C & J

ABCFIJ: D: 6+5 = 11 G: 3+4 = 7 K: 3+4 = 7 L: 5+4 = 9

M: 2+4 = 6 N: 5+1 = 6 Minimum: 6 for M & N

ABCFIJMN: D: 6+5 = 11 G: 3+4 = 7 K: 3+4 = 7 L: 5+4 = 9

Minimum: 7 for G & K

ABCFGIJKMN: D: 7+2 = 9 (shorter than previous ones!)

H: 7+4 = 11 L: 5+4 = 9 Minimum 9 for D & L

ABCDFGIJKLMN: E: 9+4 = 13 H: 9+1 = 10

Minimum: 10 for H

ABCDFGHIJKLMN: E: 10+2 = 12 Goal reached at cost 12

Shortest path from A to E: A-I-K-D-H-E