# Recommended review exercises, Chapter 12

3(3), 4(4), 7(6), <u>11(10), 12(11)</u>

In the 3rd edition of the course book, the graph treated in these problems lacks a weight on the edge between vertex A & D. It should be <u>1</u>.

<u>Solutions:</u>

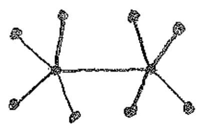
<u>3(3)</u>: (a) Is it possible for a tree to have nine vertices and degree sequence 4, 2, 2, 2, 2, 2, 2, 1, 1.

It would then have to have 8 edges but a graph with the degree sequence above must have

$$\frac{4+2+2+2+2+2+2+1+1}{2} = 9 \text{ edges.}$$

<u>NO</u>.

(b) Is it possible for a tree to have nine vertices, two of which have degree 5?

An attempt $\longrightarrow$ 

<u>NO</u>, since a tree with nine vertices will have <u>8 edges</u>, but if two vertices have degrees 5, this will require at least <u>9 edges</u> (not 10 since we can assume that the two vertices has one edge in common.)

<u>4(4)</u>: Prove that the following conditions are equivalent for a tree $\mathcal{T}$ with $n \geq 2$ vertices. (Remember: $\mathcal{T}$ has exactly $n-1$ edges.)

(i) $\mathcal{T}$ has an Eulerian trail, (ii) $\mathcal{T}$ has exactly two vertices of degree 1, (iii) $\mathcal{T}$ has exactly $n-2$ vertices of degree 2.

<u>Proof:</u> We prove (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (i). Assume (i), that $\mathcal{T}$ has an Eulerian trail $v_1v_2\dots v_m$. As it is an Eulerian trail it passes through all edges, this means that all vertices of $\mathcal{T}$ are among $v_1, v_2 \dots v_m$. As $\mathcal{T}$ is a tree (with no cycles) each vertex in $\mathcal{T}$ appears at most once in $v_1v_2\dots v_m$, this means that the vertices of $\mathcal{T}$ are exactly $v_1v_2\dots v_m$ (so $m=n$). This means that the degrees of $v_2, v_3 \dots, v_{n-1}$ will be at least 2 so that

$$\circledast \sum_{i=1}^{n} \deg(v_i) = 2\cdot|E| = 2(n-1) \Rightarrow \deg(v_1) + \underbrace{2+2+\dots+2}_{n-2} + \deg(v_n) \leq 2(n-1) \Rightarrow$$

$$\deg(v_1) + \deg(v_n) \leq 2(n-1) - 2(n-2) = 2 \Rightarrow \deg(v_1) = \deg(v_n) = 1 \text{ so (ii) holds.}$$

Now assume (ii): that $\mathcal{T}$ has exactly two vertices of degree 1. Call them $v_1$ & $v_n$. Then the degrees of $v_2, \dots, v_{n-1}$ must be at least 2. But also at most 2 since $\circledast$ holds, that means the $n-2$ vertices $v_2 \dots v_{n-1}$ has degree 2, that is (iii) holds. Similarly (iii) $\Rightarrow$ (ii).

Finally assume that (iii) hold, then (ii) hold and $\deg(v_1) = \deg(v_n) = 1$ and all other vertices has degree 2. As $\mathcal{T}$ is connected there is a unique trail from $v_1$ to $v_n$. If any vertex $v'$ would be in $\mathcal{T}$ but not in the trail from $v_1$ to $v_n$ we would have a branch point somewhere from $v_1$ to $v_n$ (since $\deg v_1 = 1$) if we want to get from $v_1$ to $v'$ — but this would require a vertex of degree 3 or higher. Hence all vertices and edges of $\mathcal{T}$ are in $v_1 \dots v_n$ and (i) holds. ▨

7(6): How many spanning trees has $K_8$? Why?

As $K_8$ has all the edges between every possible pair of vertices, a spanning tree can be formed in 7 steps after fixing one starting vertex $v_1$; we can call this "growing a spanning tree":

1. Among $\{v_2, v_3, \dots, v_8\}$ choose to add a vertex and an edge, this can be done in 7 ways. Call the chosen vertex $y_2$.

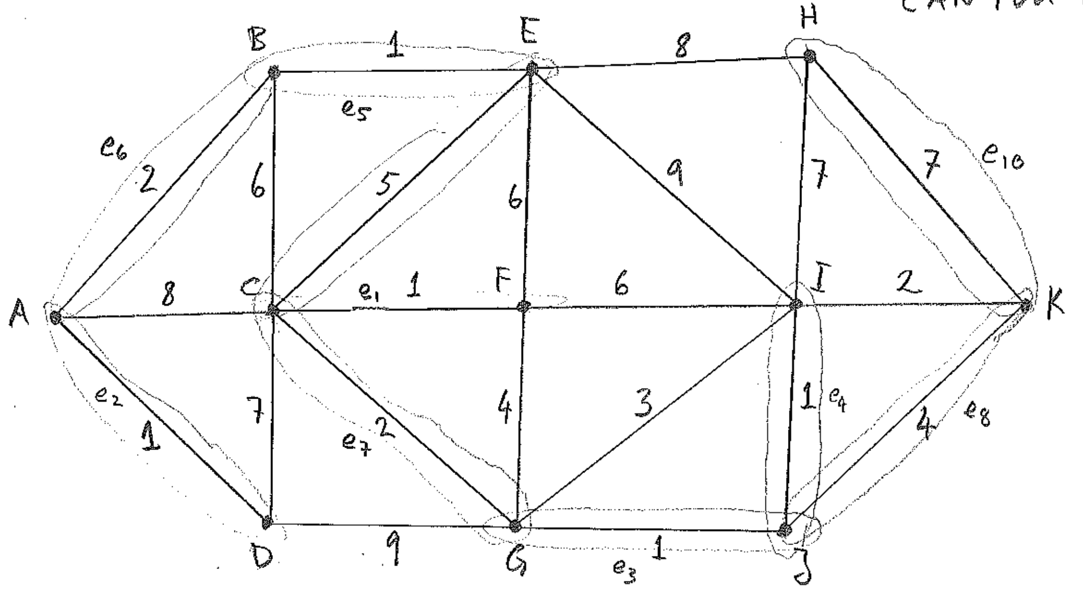
2. Now the growing spanning tree has the vertices $\{v_1, y_2\}$. From the remaining 6 vertices $\{v_2, v_3, \dots, v_8\} - \{y_2\}$ choose $y_3$, this can be done in 6 ways, connect it to one of $\{v_1, y_2\}$ this can be done in 2 ways, in total $6 \cdot 2$ ways

3. Work similarly with sets $\{v_1, y_1, y_2\}$ and $\{v_2, v_3, \dots, v_n\} - \{y_2, y_3\}$ yielding 5 and 3 giving $5 \cdot 3$ ways

4. gives $4 \cdot 4$ 5. gives $3 \cdot 5$ 6. gives $2 \cdot 6$ 7. gives $1 \cdot 7$

In total we have $7 \cdot (6 \cdot 2) \cdot (5 \cdot 3) \cdot (4 \cdot 4) \cdot (3 \cdot 5) \cdot (2 \cdot 6) \cdot 1 \cdot 7 = \underline{(7!)^2}$ spanning trees in $K_8$.

11(10): Use Kruskal's algorithm to find a minimum spanning tree of the weighted graph shown. What is the weight of a minimum spanning tree? (THIS SOLUTION CONTAINS AN ERROR CAN YOU FIND IT?)

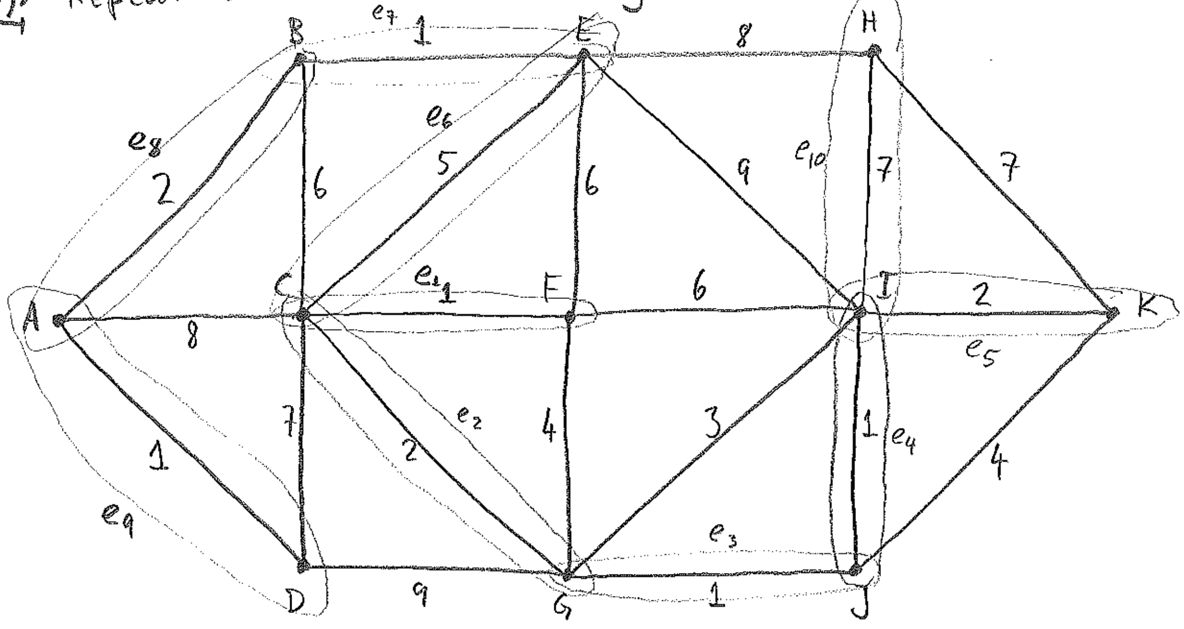



$e_1 =$ (for example) $= CF$, $e_2 = AD$, $e_3 = GJ$, $e_4 = IJ$, $e_5 = BE$

$e_6 = AB$, $e_7 = CG$, (DO NOT choose GI!), $e_8 = JK$, $e_9 = CE$, $e_{10} = HK$

Weight of the minimum spanning tree: $1+1+1+1+1+2+2+4+5+7 = \underline{25}$

12(11): Repeat exercise 11(10) using Prim's algorithm.




Choose: $e_1 = CF$, $e_2 = CG$, $e_3 = GJ$, $e_4 = IJ$, $e_5 = IK$,

$e_6 = CE$, $e_7 = BE$, $e_8 = AB$, $e_9 = AD$, $e_{10} = HI$

The weight is: $1 + 2 + 1 + 1 + 2 + 5 + 1 + 2 + 1 + 7$

$= \underline{23.}$

Can you relate this number to the error in the solution of the previous exercise?