Recommended review exercises chapter 2(2)

1(1), 3(3), 8(5), 16(10), 22(15)

Solutions:

1(1): If $A = \{x \in \mathbb{N} \mid x < 7\}$, $B = \{x \in \mathbb{Z} \mid |x-5| < 3\}$ and $C = \{2,3\}$, find $(A \oplus B) \setminus C$.

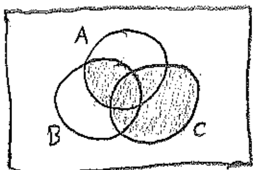
We have $A = \{x \in \mathbb{N} \mid x<7\} = \{1,2,3,4,5,6\}$ and $B = \{x \in \mathbb{Z} \mid -3 < x-5 < 3\} = \{x \in \mathbb{Z} \mid 2 < x < 8\} = \{3,4,5,6,7\}$. As $A \oplus B = A \cup B \setminus (A \cap B)$ we need to find $A \cup B$ and $A \cap B$. Clearly $A \cap B = \{3,4,5,6\}$ and $A \cup B = \{1,2,3,4,5,6,7\}$ giving $A \oplus B = \{1,2,7\}$. Subtracting C from this set gives

$$\underline{\underline{(A \oplus B) \setminus C = \{1,2,7\} \setminus \{2,3\} = \{1,7\}}}.$$

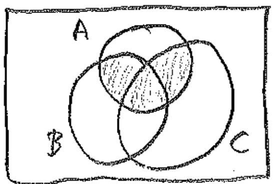
3(3): Let A, B, C be sets. Are the following statements true or false? In each case provide a proof or exhibit a counterexample.

(a) $A \cap B = A$ if and only if $A \subseteq B$.
This is true — so now we prove it. It is $A \cap B = A \Leftrightarrow A \subseteq B$.
$\Rightarrow$) Assume $A \cap B = A$ and assume $x \in A$. Then as $A = A \cap B$ we also have $x \in A \cap B$ and so $x \in B$. But then we have shown the implication $x \in A \Rightarrow x \in B$. This means exactly that B contains A, that is $A \subseteq B$.
$\Leftarrow$) Assume $A \subseteq B$. Then $A \cap A \subseteq A \cap B \Rightarrow A \subseteq A \cap B$, but since $A \cap B \subseteq B$ we conclude $A = A \cap B$. The proof is complete.

(b) $(A \cap B) \cup C = A \cap (B \cup C)$. Let us consider Venn diagrams:

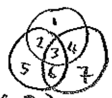


$(A \cap B) \cup C$



$A \cap (B \cup C)$

Clearly the sets seem different. We can construct a counter-example by setting



$A = \{1,2,3,4\}$
$B = \{2,3,5,6\}$ and $C = \{3,4,6,7\}$

For these sets we have

$$(A \cap B) \cup C = \{2,3,4,6,7\} \neq A \cap (B \cup C) = \{2,3,4\}$$

so $(A \cap B) \cup C = A \cap (B \cup C)$ is a <u>false</u> statement as shown by the exhibited counterexample.

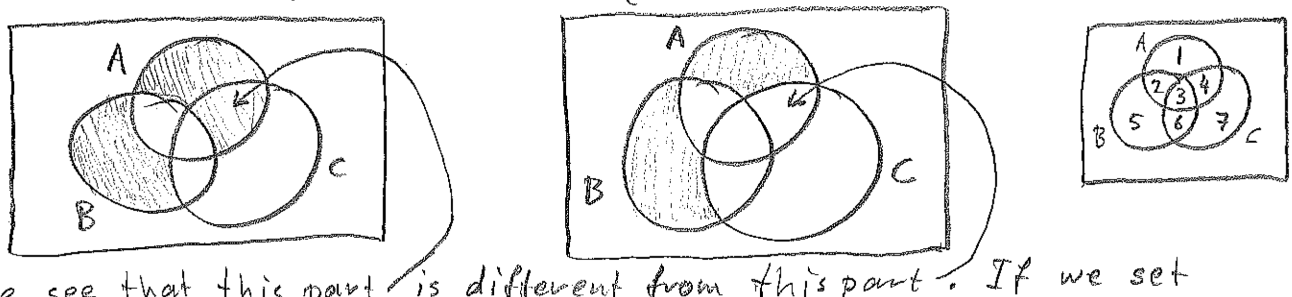
(c) $A \cap B = \emptyset \rightarrow A \neq B$. This is <u>false</u> since $A = B = \emptyset$ have $A \cap B = \emptyset$ but $A = B$.

(Is this true if one of A & B (or both) are nonempty?)

8(5): Let A, B, C, D be sets.

(a) Give an example showing that the statement $A \oplus (B \setminus C) = (A \oplus B) \setminus C$ is false.
To find such an example we utilize Venn diagrams

$A \oplus (B \setminus C)$ $\qquad$ $(A \oplus B) \setminus C$



We see that this part is different from this part. If we set $A = \{1,2,3,4\}$, $B = \{2,3,5,6\}$, and $C = \{3,4,6,7\}$ we have

$$A \oplus (B \setminus C) = \{1,4,5\} \neq (A \oplus B) \setminus C = \{1,5\}.$$

(b) Prove that the statement $A \subseteq C,\ B \subseteq D \Rightarrow A \times B \subseteq C \times D$ is true.

Proof: To show that $A \times B \subseteq C \times D$ we must show the implication $(a,b) \in A \times B \Rightarrow (a,b) \in C \times D$ where a, b denotes elements in A, B.
So suppose $(a,b) \in A \times B$. By definition this means $a \in A$ and $b \in B$ and since $A \subseteq C$ and $B \subseteq D$ we also have $a \in C$ and $b \in D$. But this means, by definition, $(a,b) \in C \times D$, hence the implication is shown which completes the proof.

16(10): Define a relation R on $\mathbb{Z}$ by $aRb \Leftrightarrow 2a+5b$ is a multiple of 7.

(a) Prove that R is an equivalence relation on $\mathbb{Z}$.

Proof: We need to prove three things: Reflexivity, Symmetry & Transitivity.

Reflexivity: Show that each $a \in \mathbb{Z}$ is related to itself, that is that $aRa \Leftrightarrow$ $2a+5a$ is a multiple of 7. But this is clear since $2a+5a = 7 \cdot a$.

Symmetry: Show that if aRb, then bRa. So suppose aRb, then $2a+5b = 7q$, for some q, but then also $-2a-5b = -7q \Rightarrow$ $7a-2a+7b-5b = 7a+7b-7q \Rightarrow 5a+2b = 7(a+b-q) \Rightarrow 2b+5a = 7(a+b-q)$ so that $2b+5a$ is a multiple of 7 showing that bRa which completes the proof of symmetry.

Transitivity: Show that if aRb and bRc, then aRc. Suppose that $2a+5b = 7 \cdot q_1$ and $2b+5c = 7q_2$. Then $2a+5c = 7q_1 - 5b + 7q_2 - 2b$; so that $2a+5c = 7(q_1+q_2-b)$ and so $2a+5c$ is a multiple of 7 completing the proof of transitivity.

(b) Is R a partial order? A partial order has reflexivity & transitivity but anti-symmetry, aRb and bRa would mean $a=b$ but we do not have that, for example $1R7$ and $7R1$ but $7 \neq 1$.

22 (15): Let $A = \{1, 2, 4, 6, 8\}$ and define $a \preceq b \Leftrightarrow \frac{b}{a}$ is an integer.

(a) Prove that $\preceq$ is a partial order on A.

Proof: We must show three things: Reflexivity, anti-symmetry, transitivity.

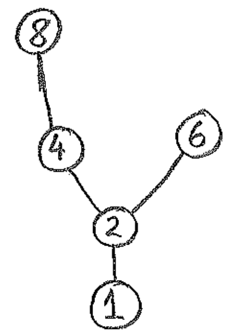
Reflexivity: Show that $a \preceq a$ for all $a \in A$. Indeed $\frac{a}{a} = 1$ which is an integer for all $a \in A$ so that $a \preceq a$ showing reflexivity.

anti-symmetry: If $a \preceq b$ and $b \preceq a$, show that $a = b$. Well if $\frac{b}{a} = q_1$ for some integer q_1 and $\frac{a}{b} = q_2$ for some integer q_2 then $q_1 \cdot q_2 = \frac{b}{a} \cdot \frac{a}{b} = 1$. Since both q_1 & q_2 are positive we must have both q_1 and q_2 being 1 itself, which shows $\frac{b}{a} = 1 \Leftrightarrow b = a$, which completes the proof of anti-symmetry.

Transitivity: Show that $a \preceq b$ and $b \preceq c$ gives $a \preceq c$, so suppose $\frac{b}{a} = q_1$ and $\frac{c}{b} = q_2$ for integers q_1, q_2. But then $\frac{c}{a} = \frac{c}{b} \cdot \frac{b}{a} = q_2 \cdot q_1 = q_3$, another integer, so that indeed $a \preceq c$. Transitivity is shown.

(b) Draw the Hasse diagram for $\preceq$.

Clearly $1 \preceq a$ for all $a \in A$, further $2 \preceq 4 \preceq 8$ and $2 \preceq 6$. No other relations can be found, we do not have $4 \preceq 6$ or $6 \preceq 8$.



(c) 6 and 8 are maximal elements. None of them is a maximum (since they are distinct.)

1 is a minimal element that is also a minimum.