Recommended review exercises, Chapter 3(3)

2(2), 4(3), 8(4), 13(8), 14(9)

Solutions:

2(2): Let $f = \{(1,2),(2,3),(3,4),(4,1)\}$ and $g = \{(1,3),(2,1),(3,4),(4,2),(5,1)\}$.

Find f^{-1} and $g \circ f$. Is g one-to-one? Explain.

We note that this assignment uses set notation to define functions f & g. The above notation (with sets) is what we commonly think of as this: $f(1)=2$, $f(2)=3$, $f(3)=4$, $f(4)=1$ and $g(1)=3$, $g(2)=1$, $g(3)=4$, $g(4)=2$, and $g(5)=1$.

To find f^{-1} we simply form $f^{-1} = \{(2,1),(3,2),(4,3),(1,4)\}$ and this also forms a function so f is one-to-one. The inverse function is given by this set notation but we could also write $f^{-1}(2)=1$, $f^{-1}(3)=2$, $f^{-1}(4)=3$, and $f^{-1}(1)=4$.

The function $g \circ f$ is "the composition of g and f". It works like this: first apply f, then apply g to what comes out of f. Looking at the definitions above we see that

$g \circ f(1) = g(f(1)) = g(2) = 1$, $g \circ f(2) = g(f(2)) = g(3) = 4$,
$g \circ f(3) = g(f(3)) = g(4) = 2$, $g \circ f(4) = g(f(4)) = g(1) = 3$.

This can also be written in set notation:

$$g \circ f = \{(1,1),(2,4),(3,2),(4,3)\}.$$

The function g is not one-to-one because there are more than one value that maps to 1, we have $g(2) = g(5) = 1$.

4(3): Define $f: \mathbb{Z} \to \mathbb{Z}$ by $f(x) = 2x^3 + x$. Show that f is one-to-one, but not onto.

To show that f is one-to-one we can observe that f is increasing, this means that $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. This means that if $x_1 \neq x_2$ then either $x_1 < x_2$ or $x_1 > x_2$ so either $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$ that is $f(x_1) \neq f(x_2)$ that is the implication $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ is always true for all x_1, x_2, this is exactly the meaning of f being one-to-one.

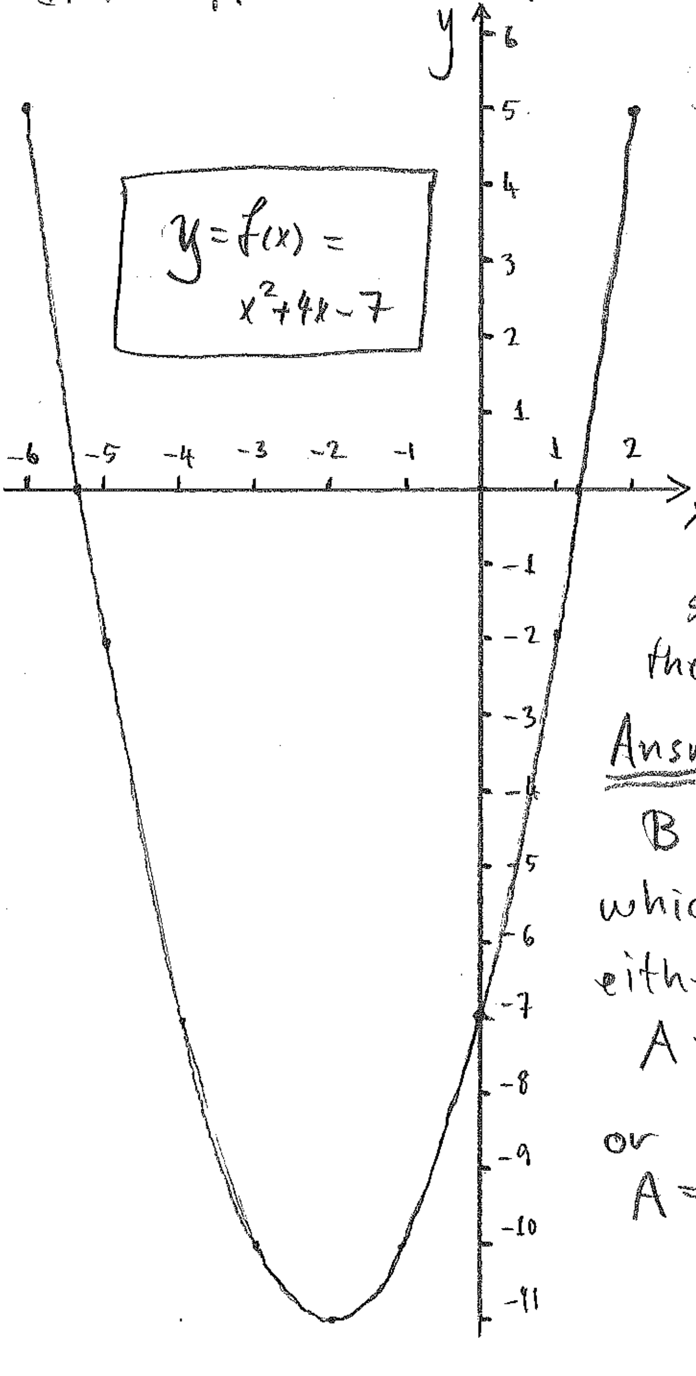
To show that f is not onto it is enough to show that there is a value that f does not assume. 1 is such a value. (Why?)

8(4): Can a function be a reflexive relation? Explain.

Formally, a function is a relation which is a set of ordered pairs. That a relation is reflexive means that xRx for all x, and this can indeed also be a function namely the function that has $f(x) = x$ for all x since this means that the function consists of all elements on the form (x, x). So the answer is <u>yes</u>.

13(8): Find subsets of $\mathbb{R}$, A and B, with A as large as possible, such that $f: A \to B$ defined by $f(x) = x^2 + 4x - 7$ is one-to-one and onto.

It is helpful to draw a pictorial representation of f:



We can state and prove mathematical theorems that clearly describes the various maximal sets A & B. However it is more suitable for us to appeal to our geometrical understanding and just state the answer based on the figure.

<u>Answer:</u> The maximal set B is $\{y \in \mathbb{R} \mid y \geq -11\}$ which is obtained from either

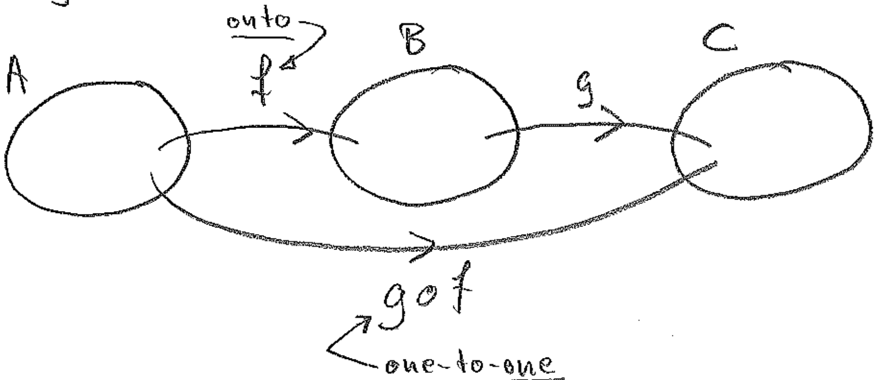
$A = \{x \in \mathbb{R} \mid x \leq -2\}$

or

$A = \{x \in \mathbb{R} \mid x \geq -2\}$.

14 (9): Suppose $f: A \to B$ and $g: B \to C$ are functions. If $g \circ f$ is one-to-one and f is onto, show that g is one-to-one.

Proof: We wish to prove that $y_1 \neq y_2 \Rightarrow g(y_1) \neq g(y_2)$, or, equivalently $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$, based on the facts that $g \circ f$ is one-to-one and f is onto.



Choose y_1 and y_2 arbitrarily in B such that $g(y_1) = g(y_2)$. Since $f: A \to B$ is onto there exists x_1 and x_2 in A such that $f(x_1) = y_1$ and $f(x_2) = y_2$. This then gives

$$g(f(x_1)) = g(f(x_2)) \Leftrightarrow g \circ f(x_1) = g \circ f(x_2).$$

But since $g \circ f$ is one-to-one we conclude $x_1 = x_2$ which gives $y_1 = f(x_1) = f(x_2) = y_2$, that is $y_1 = y_2$. Since we have shown that the implication

$$f(y_1) = f(y_2) \Rightarrow y_1 = y_2$$

is true, we have shown that f is one-to-one which completes the proof.