Recommended review exercises Chapter 5

2(1), 3(2), 13(5), 19(8), 23(9)

Solutions:

2(1): Using mathematical induction, show that $\sum_{i=1}^{n} 3^{i-1} = \frac{3^n-1}{2}$ for all $n \geq 1$.

Proof: The induction proof first requires us to check that the statement is true for $n=1$, when we only have one term in the sum. Let us check, is it true that

$$\sum_{i=1}^{1} 3^{i-1} = \frac{3^1-1}{2} \Leftrightarrow 3^{1-1} = \frac{2}{2}$$

well yes it is true $3^0 = 1 = \frac{2}{2}$.

Now to take the induction step we must show that if the statement is true for $n=k$ (a certain n), then it also becomes true for $n=k+1$ (the next one.) In effect we are showing that the implication $S_k \to S_{k+1}$ is true.

So suppose $\sum_{i=1}^{k} 3^{i-1} = \frac{3^k-1}{2}$ for a certain k. Can we show $\sum_{i=1}^{k+1} 3^{i-1} = \frac{3^{k+1}-1}{2}$? Wait a second... think... We see

$$\underbrace{\sum_{i=1}^{k} 3^{i-1}}_{\frac{3^k-1}{2}} + 3^{k+1-1} = \frac{3^k-1}{2} + 3^k = \frac{3^k-1+2\cdot 3^k}{2} = \frac{3\cdot 3^k-1}{2} = \underline{\underline{\frac{3^{k+1}-1}{2}}}$$

It was here that we used that the statement was true for $n=k$

and indeed we see that $\sum_{i=1}^{k+1} 3^{i-1} = \frac{3^{k+1}-1}{2}$ if $\sum_{i=1}^{k} 3^{i-1} = \frac{3^k-1}{2}$.

This means that if the statement holds for $n=k$, it also holds for $n=k+1$. If the statement is called S_n, we have therefor shown

S_1 true $\Rightarrow$ S_2 true $\Rightarrow$ S_3 true $\Rightarrow$ and on and on...

according to the induction axiom we have

$\forall n \geq 1 : S_n$ which we wanted to prove.

You MUST include this part in the proof

3(2): Using mathematical induction, show that $\left(1-\frac{1}{2}\right)^n \geq 1-\frac{n}{2}$ for all $n \geq 1$.

Proof: Check $n=1$: Is it true that $\left(1-\frac{1}{2}\right)^1 \geq 1-\frac{1}{2}$?, well, yes, since $\left(1-\frac{1}{2}\right)^1 = \frac{1}{2} \geq \frac{1}{2} = 1-\frac{1}{2}$.

Now show that if the statement is true for $n=k$ it is also true for $n=k+1$: Suppose it is true for $n=k$ and show that this leads to it being true for $n=k+1$.

Suppose: $\left(1-\frac{1}{2}\right)^k \geq 1-\frac{k}{2}$

Show: $\left(1-\frac{1}{2}\right)^{k+1} \geq 1-\frac{k+1}{2}$

wait a second... think... compute... compare...

$$\left(1-\frac{1}{2}\right)^{k+1} = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)^k = \frac{1}{2}\cdot\left(1-\frac{1}{2}\right)^k \geq \frac{1}{2}\cdot\left(1-\frac{k}{2}\right) \geq \frac{1}{2}-\frac{1}{2}\cdot\frac{k}{2} \geq$$

Here we used what we supposed

change $\frac{1}{2}$ to 1

$$\frac{1}{2}-\frac{k}{2} = 1-\frac{1}{2}-\frac{k}{2} = 1-\left(\frac{k}{2}+\frac{1}{2}\right) = 1-\frac{k+1}{2}$$ which was the goal!

So the statement being true for $n=k$ implies the statement being true for $n=k+1$. We have $S_k \Rightarrow S_{k+1}$ for all k.

In conclusion

S_1 true $\Rightarrow$ S_2 true $\Rightarrow$ S_3 true $\Rightarrow$ ... and so on giving that S_n is true for all n according to the induction axiom. The proof is complete.

You must include this part of the proof.

(Remark: this is actually much simpler without mathematical induction... why?)

23(9): Solve the recurrence relation $a_n = 5a_{n-1} - 4a_{n-2}$ $n \geq 2$ given that $a_0 = -3$ and $a_1 = 6$.

Characteristic equation: $x^2 = 5x-4 \Leftrightarrow x^2-5x+4=0 \Leftrightarrow x = 2.5 \pm \sqrt{6.25-4} \Leftrightarrow$ $x = 2.5 \pm 1.5 \Leftrightarrow x=-1 \vee x=4$, so the solution can be written

$$a_n = c_1\cdot(-1)^n + c_2\cdot 4^n.$$

Using that $a_0=-3$ and $a_1=6$ we get the following system of equations:

$$\begin{cases} a_0 = c_1(-1)^0 + c_2\cdot 4^0 = -3 \\ a_1 = c_1\cdot(-1)^1 + c_2\cdot 4^1 = 6 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 = -3 \\ -c_1 + 4c_2 = 6 \end{cases} \overset{(+)}{\Leftrightarrow} \begin{cases} c_1 = -3-c_2 = -3-\frac{3}{5} = -\frac{15}{5}-\frac{3}{5} \\ 5c_2 = 3 \Leftrightarrow c_2 = \frac{3}{5} \end{cases}$$

So that $c_1 = -\frac{18}{5}$ and $c_2 = \frac{3}{5}$

and $a_n = c_1(-1)^n + c_2\cdot 4^n = \underline{\underline{-\frac{18}{5}\cdot(-1)^n + \frac{3}{5}4^n}}$

13(5): Let a_n be recursively defined by $a_1=0$, $a_2=\frac{1}{3}$, and, for $k\geq 1$, $a_{k+2}=\frac{1}{2}(a_k+a_{k+1})$. Prove that $a_n=\frac{2}{9}\left(1-\left(-\frac{1}{2}\right)^{n-1}\right)$ for all $n\in\mathbb{N}$.

Proof: Strong induction over n: Give the statement that $a_n=\frac{2}{9}\left(1-\left(-\frac{1}{2}\right)^{n-1}\right)$ the name S_n. We shall prove $\forall n\in\mathbb{N}: S_n$.

First check S_1 and S_2:

S_1: $a_1=\frac{2}{9}\left(1-\left(-\frac{1}{2}\right)^{1-1}\right)=0$, this is true since $\left(-\frac{1}{2}\right)^0=1$.

S_2: $a_2=\frac{2}{9}\left(1-\left(-\frac{1}{2}\right)^{2-1}\right)=\frac{1}{3} \Leftrightarrow \frac{2}{9}\left(1-\left(-\frac{1}{2}\right)\right)=\frac{1}{3} \Leftrightarrow \frac{2}{9}\cdot\frac{3}{2}=\frac{1}{3}$ which is true.

Now assume, that all $S_1, S_2, S_3, \ldots, S_k$ are true, $k\geq 2$, and prove that S_{k+1} is true. The assumption will imply

S_k: $a_k=\frac{2}{9}\left(1-\left(-\frac{1}{2}\right)^{k-1}\right)$ and S_{k-1}: $a_{k-1}=\frac{2}{9}\left(1-\left(-\frac{1}{2}\right)^{k-2}\right)$

from this we have to show that S_{k+1}: $a_{k+1}=\frac{2}{9}\left(1-\left(-\frac{1}{2}\right)^{k}\right)$ is true.

wait a second... think... compute...

By the recursive definition of all a_n we have

$$a_{k+1}=\frac{1}{2}(a_{k-1}+a_k)\underset{\text{by } S_k \,\&\, S_{k-1}}{=}\frac{1}{2}\cdot\left(\frac{2}{9}\left(1-\left(-\frac{1}{2}\right)^{k-2}\right)+\frac{2}{9}\left(1-\left(-\frac{1}{2}\right)^{k-1}\right)\right)=$$

$$=\frac{2}{9}\left(\frac{1}{2}-\frac{1}{2}\left(-\frac{1}{2}\right)^{k-2}+\frac{1}{2}-\frac{1}{2}\left(-\frac{1}{2}\right)^{k-1}\right)=\frac{2}{9}\left(1+\left(-\frac{1}{2}\right)^{k-1}+\left(-\frac{1}{2}\right)^{k}\right)=$$

$\frac{2}{9}\left(1-2\cdot\left(-\frac{1}{2}\right)^k+\left(-\frac{1}{2}\right)^k\right)=\frac{2}{9}\left(1-\left(-\frac{1}{2}\right)^k\right)$ which is what we needed, this means that S_{k+1} is true.

In conclusion:

S_1 & S_2 are true

S_1 & S_2 & ... $S_k \Rightarrow S_{k+1}$ for all $k\geq 2$

this means that we have S_1 true $\Rightarrow S_2$ true $\Rightarrow S_3$ true $\Rightarrow\ldots$ and so on. According to the induction axiom (strong version) we conclude $\forall n\geq 1: S_n$ which completes the proof.

Alternative solution: Solve the recurrence relation

$$a_1=0,\quad a_2=\frac{1}{3},\quad a_{n+2}=\frac{1}{2}(a_{n+1}+a_n)$$

(Characteristic equation $x^2=\frac{1}{2}(x+1)$ ... $x=1 \vee x=-\frac{1}{2}$

$$a_n=C_1\cdot(1)^n+C_2\cdot\left(-\frac{1}{2}\right)^n \text{ etc...}$$)

19 (8):

(a) Define the Fibonacci sequence.

Definition: $f_0 = 1$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$ for all $n \geq 0$.

(b) Is it possible for three successive terms in the Fibonacci sequence to be odd?

No, since all numbers fulfill $f_{n+2} = f_{n+1} + f_n$, if two successive terms would be odd then we would always have the next one even, like this: $f_n = \text{odd}$ $f_{n+1} = \text{odd}$ $\Rightarrow f_{n+2} = f_{n+1} + f_n$

$= \text{odd} + \text{odd} = \underline{\text{even}}.$

(c) Is it possible for two succesive terms in the Fibonacci sequence to be even?

No, we can even state and prove a description of the pattern of the oddness and evenness of f_n, we will not give a complete proof here but describe the argument.

Description of the terms of the Fibonacci Sequence: The terms are: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

the pattern is

odd, odd, even, odd, odd, even, odd, odd, even, ...

that is two odd numbers followed by an even, followed by two odd, followed by one even, and so on.

This pattern is maintained by the fact that each number is the sum of the two previous ones. Convince yourself of the generality of this argument by studying the sequence.