Recommended review exercises Chapter 6

2(2), 6(5), 7(7), 8(8)

2(2): Using the Principle of Inclusion-Exclusion, find the number of integers in $\{1, 2, 3, \dots, 1999, 2000\}$ that are divisible by at least one of 2, 3, 5, 7.

For the sake of clarity, we introduce the sets

$$\mathcal{U} = \{1, 2, 3, \dots, 1999, 2000\}$$

and A = the set of integers in $\mathcal{U}$ that are divisible by 2 = $\{x \in \mathcal{U} ; 2|x\}$,

and $B = \{x \in \mathcal{U} ; 3|x\}$, and $C = \{x \in \mathcal{U} ; 5|x\}$, and $D = \{x \in \mathcal{U} ; 7|x\}$.

Our task is to find $|A \cup B \cup C \cup D|$ but since the sets A, B, C, D are not disjoint (for eample $6 = 2 \cdot 3 \in A \cap B$ and $35 = 7 \cdot 5 \in C \cap D$) we need to use the principle of incl./excl. for four sets:

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D| \quad (1)$$

All of these sets are of the form "all multiples of d in $\mathcal{U}$", where $d = 2, 3, 5, 7$ (for A, B, C, D), $d = 6, 10, 14, 15, 21, 35$ (for $A \cap B, A \cap C, A \cap D, B \cap C, B \cap D, C \cap D$) so the procedure is the same for all sets, for example $A = \{2 \cdot k ; k = 1, 2, 3, \dots, 1000\}$ which means that $|A| = 1000$. Further $B = \{3 \cdot k ; k = 1, 2, 3, \dots, 666\}$ which means that $|B| = 666$. Similarly $C = \{5 \cdot k ; k = 1, 2, 3, \dots, 400\} \Rightarrow |C| = 400$, $D = \{7 \cdot k ; k = 1, 2, 3, \dots, 285\} \Rightarrow |D| = 285$.

Trudging on we have $A \cap B = \{6 \cdot k ; k = 1, 2, 3, \dots, 333\} \Rightarrow |A \cap B| = 333$, $A \cap C = \{10 \cdot k ; k = 1, 2, 3, \dots, 200\} \Rightarrow |A \cap C| = 200$, $A \cap D = \{14 \cdot k ; k = 1, 2, \dots, 142\} \Rightarrow |A \cap D| = 142$, $B \cap C = \{15 \cdot k ; k = 1, 2, 3, \dots, 133\} \Rightarrow |B \cap C| = 133$, $B \cap D = \{21 \cdot k ; k = 1, 2, \dots 95\} \Rightarrow |B \cap D| = 95$, $C \cap D = \{35 \cdot k ; k = 1, 2, \dots, 57\} \Rightarrow |C \cap D| = 57$. Continuing with the sets formed from three factors we get $A \cap B \cap C = \{30 \cdot k ; k = 1, 2, \dots, 66\} \Rightarrow |A \cap B \cap C| = 66$, $A \cap B \cap D = \{42 \cdot k ; k = 1, 2, \dots, 47\} \Rightarrow |A \cap B \cap D| = 47$, $A \cap C \cap D = \{70 \cdot k ; k = 1, 2, \dots, 28\} \Rightarrow |A \cap C \cap D| = 28$, $B \cap C \cap D = \{105 \cdot k ; k = 1, 2, \dots, 19\} \Rightarrow |B \cap C \cap D| = 19$, and finally $A \cap B \cap C \cap D = \{210 \cdot k ; k = 1, 2, 3, \dots, 9\} \Rightarrow |A \cap B \cap C \cap D| = 9$. Combining all these findings into (1) we get

$$|A \cup B \cup C \cup D| = 1000 + 666 + 400 + 285 - 333 - 200 - 142 - 133 - 95 - 57 + 66 + 47 + 28 + 19 - 9 = 1542$$

6(5): Seventy cars in total. 30 stereosystems, 30 AC, 40 sun roofs. 30 have at least two of these options, 10 have all three.
How many have at least one of these options? How many have exactly one?

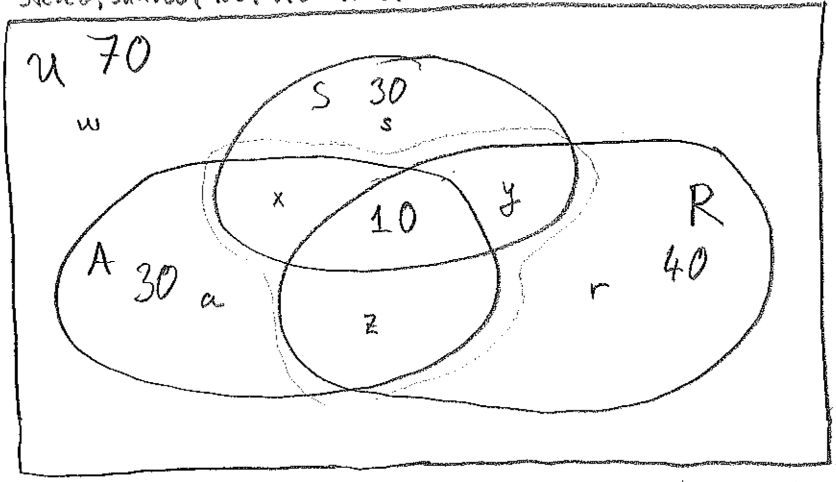
For clarity introduce letters to denote the various quantities:

S = the cars with stereos, $|S| = 30$
A = the cars with AC, $|A| = 30$
R = the cars with sun roofs, $|R| = 40$

$$|(S \cap A) \cup (A \cap R) \cup (S \cap R)| = 30$$

$$|S \cap A \cap R| = 10 \qquad \mathcal{U} = \text{all cars}, \ |\mathcal{U}| = 70$$

Introduce further notation: a = cars with only AC, s = cars with only stereo, r = cars with only sun roof, x = cars with stereo, AC not sun roof, y = cars with stereo, sunroof NOT AC and z = cars with AC, sun roof not stereo, w = cars with nothing



Reading the text we can write down the following equations:

$$x+y+z+10 = 30 \iff x+y+z = 20$$

$$\left.\begin{aligned} x+y+s &= 20 \\ a+x+z &= 20 \\ + \quad r+y+z &= 40 \end{aligned}\right\} \text{sum these}$$

$$2x+2y+2z+a+s+r = 20+20+40 \Rightarrow$$

$$\underbrace{x+y+z}_{20} + x+y+z+a+s+r = 80 \Rightarrow x+y+z+a+s+r = 60 \Rightarrow$$

$$a+x+s+y+r+z+10 = 70$$

So cars with at least one of the three options is $\underline{\underline{70}}$

Cars with exactly one option is

$$a+s+r = 60-(x+y+z) = 60-20 = \underline{\underline{40}}$$

7(7): The Strong form of the Pigeonhole Principle: If n objects are put into m boxes and $n > m$ then some box must contain at least $\lceil \frac{n}{m} \rceil$ objects.

Example: If we have 50 flowers and distribute them among Amita, Imane, Melisa, Noora, Sara, and Selma, then one of them will receive at least $\lceil \frac{50}{6} \rceil = 9$ flowers.

8(8): Show that among 18 arbitrarily chosen integers there must exist two whose difference is divisible by 17.

Proof: Every integer belongs to one of the 17 congruence classes modulo 17, this means that we can assign, to each of of the arbitrarily chosen integers the congruence class to which it belongs. The 17 congruence classes then are the boxes and the 18 integers are the objects, and according to the Pigeonhole Principle there must be at least one congruence class (box) that will have two integers (objects) assigned to it, that is, two of the 18 arbitrarily chosen integers will belong to the same congruence class which means exactly that their difference is divisible by 17. The proof is complete.