Recommended review exercises Chapter 7

4(3), 14(7), 19(9), 20(10), 22(11), 26 (NA)

Solutions:

4(3): How many permutations of the letters a, b, c, d, e, f contain at least one of the patterns aeb and bcf

For clarity we introduce two sets

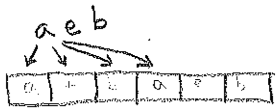
A = the set of all permutations of abcdef containing aeb

B = " bcf

Our task is to find $|A \cup B|$ which is (according to the Principle of inclusion/exclusion) equal to the number $|A| + |B| - |A \cap B|$.

We start with $|A|$. The number of elements in A is equal to the number of possibilities in constructing an element in A, which can be determined by the Multiplication rule:

Step 1. Choose what position a has, this can be done in 4 ways



Step 2. choose where c is to be put: 3 ways

Step 3. Choose where d is to be put: 2 ways

Step 4. Choose where f is to be put: 1 way.

Total number of ways $= 4 \cdot 3 \cdot 2 \cdot 1 = 24 = |A|$. B is treated in an exactly analogous way, so we also have $|B| = 24$.

The set $A \cap B$ consists of those permutations that contain both aeb and bcf, since b only occurs once this means that these are the permutations that contain aebcf, so once the position of a is fixed all the other letters' positions are determined. It is obvious that only the permutations

aebcfd and daebcf

are the elements in $A \cap B$ so that $|A \cap B| = 2$.

In conclusion

$$|A \cup B| = 24 + 24 - 2 = \underline{\underline{46}}$$

14(7): Frank wants to buy 12 muffins and he finds 7 different types available. In how many ways can he make his selection?

Frank can indicate his choice by dropping $r = 12$ (identical) marbles into $n = 7$ boxes, one box for each type of muffin, then we can graphically illustrate by a sequence of ones and zeroes:

$10000111000100110001 \leftarrow$ This depicts Frank buying 4 of the first kind, 3 of the fourth kind and so on.

The number of such strings of ones and zeroes is

$$\binom{n+r-1}{r} = \binom{7+12-1}{12} = \binom{18}{12} = \frac{18\cdot 17\cdot 16\cdot 15\cdot 14\cdot 13}{6\cdot 5\cdot 4\cdot 3\cdot 2} = 17\cdot 4\cdot 3\cdot 7\cdot 13$$

19(9): Find the coefficient of x^5 in the binomial expansion of $(x-2x^{-2})^{20}$. Do not simplify.

According to the Binomial Theorem we have $(x-2x^{-2})^{20} = \sum_{k=0}^{20} \binom{20}{k} x^k (-2x^{-2})^{20-k}$

when do we have the term $\binom{20}{k} x^k (-2x^{-2})^{20-k}$ having the power 5 on x? $x^k(-2x^{-2})^{20-k} = x^{k-40+2k}\cdot(-2)^{20-k} = x^{3k-40}\cdot(-2)^{20-k}$

$3k-40 = 5 \Leftrightarrow 3k = 45 \Leftrightarrow k = 15$ so the term with x^5 is $\binom{20}{15}\cdot(-2)^{20-15}\cdot x^5$ and the coefficient thus is $\binom{20}{15}(-2)^5$ $\left(= -32\binom{20}{5}\right)$

20(10): Find the coefficient of x^{-6} in $\left(16x^2 - (2x)^{-1}\right)^{12}$. Simplify the answer.

Again we apply the Binomial Theorem giving

$$\left(16x^2 + (-2x)^{-1}\right)^{12} = \sum_{k=0}^{12} \binom{12}{k} (16x^2)^k \cdot \left((-2x)^{-1}\right)^{12-k}.$$

Then we seek the term with the exponent -6 for x, studying one term ...

$$\binom{12}{k} 16^k\cdot x^{2k}\cdot(-2)^{k-12}\cdot x^{k-12} = \binom{12}{k}\cdot 16^k\cdot(-2)^{k-12}\cdot x^{3k-12},$$

the exponent that was saught was -6 so seek the k for which $3k-12 = -6 \Leftrightarrow 3k = 6 \Leftrightarrow k = 2$. For this k we get the coefficient being

$$\binom{12}{2}\cdot 16^2\cdot\frac{1}{(-2)^{10}} = \frac{12\cdot 11}{2}\cdot\frac{16^2}{2^{10}} = \frac{12\cdot 11\cdot 2^8}{2^{11}} = \frac{12\cdot 11}{2^3} = \frac{4\cdot 3\cdot 11}{4\cdot 2} = \underline{\underline{\frac{33}{2}}}$$

22(11): (a) Give a verbal argument for the truth of the identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$\binom{n}{k}$ is the number of ways to choose k elements from a set containing n elements. To see that it is also equal to $\binom{n-1}{k-1} + \binom{n-1}{k}$ we assume that $n=5$ and $k=2$. The argument will be exactly analogous in the general case. We can even call the elements a, b, c, d, e. Fix one element, a. $\binom{5}{2}$ is the number of ways to choose two from $\{a,b,c,d,e\}$. But this number of ways can calculated by first computing the number of ways to choose two from $\{a,b,c,d,e\}$ INCLUDING a, which is $\binom{4}{1} = \binom{n-1}{k-1}$, and then computing the number of ways of choosing two from $\{a,b,c,d,e\}$ NOT INCLUDING a, which is $\binom{4}{2} = \binom{n-1}{k}$, and then since these are disjoint, by the addition rule we get $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$, or, in general, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

(b) If $n \geq k+2 \geq 2$ show that $\binom{n}{k} - \binom{n-2}{k} - \binom{n-2}{k-2}$ is even.

Using the formula above we have

$$\binom{n}{k} - \binom{n-2}{k} - \binom{n-2}{k-2} = \underbrace{\binom{n-1}{k-1}}_{\binom{n-2}{k-1}+\binom{n-2}{k-2}} + \underbrace{\binom{n-1}{k}}_{\binom{n-2}{k}+\binom{n-2}{k-1}} - \binom{n-2}{k} - \binom{n-2}{k-2} =$$

$$= \binom{n-2}{k-1} + \cancel{\binom{n-2}{k-2}} + \cancel{\binom{n-2}{k}} + \binom{n-2}{k-1} - \cancel{\binom{n-2}{k}} - \cancel{\binom{n-2}{k-2}} =$$

$2 \cdot \binom{n-2}{k-1}$ which is an even number.

26(NA): A fair coin is tossed n times. Prove that the probability of getting an odd number of heads is $\frac{1}{2}$.

Proof: Induction over n: Name the statement "The probability of getting an odd number of heads when tossing a fair coin n times is $\frac{1}{2}$" S_n.

S_1 = "the probability of getting an odd nr of heads when tossing a coin once is $\frac{1}{2}$"

is obviously true.

Now assume that S_k is true and prove that S_{k+1} follows.

$$P(\text{odd nr of heads}, k+1) =$$

$$P(\text{heads at } k+1) \cdot \underbrace{P(\text{odd nr of heads}, k)}_{=1/2} + P(\text{tails at } k+1) \cdot \underbrace{P(\text{odd nr of heads}, k)}_{=\frac{1}{2} \text{ according to } S_k}$$

$$= P(\text{heads at } k+1) \cdot \frac{1}{2} + P(\text{tails at } k+1) \cdot \frac{1}{2}$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

which means S_{k+1} is true.

In conclusion:

S_1 true $\Rightarrow S_2$ true $\Rightarrow S_3$ true $\Rightarrow \ldots$

and so on, according to the Principle of Mathematical Induction we conclude

$$\forall n \geq 1 : S_n \quad \text{QED.}$$