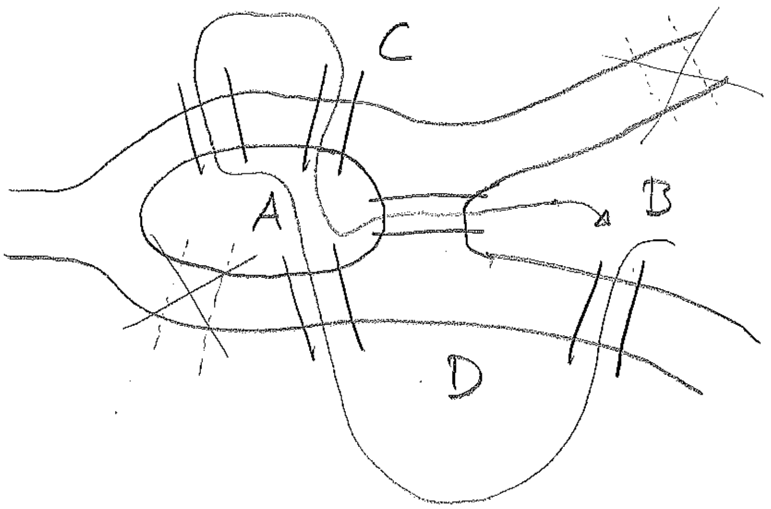
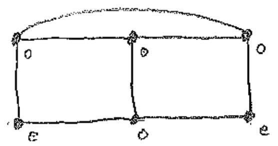
Recommended review exercises Chapter 9
1(1), 4(4), 6(5), 12(8), 15(9)

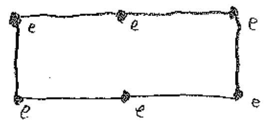
1(1): In the Königsberg Bridge Problem, a tragic fire destroys the bridge from B to C and also one of the bridges from A to D. Draw a graph representing the new situation. Show that it is now possible for someone to start on land mass B and walk over each of the bridges exactly once, returning to B again.



4(4): (a) Draw a graph with six vertices at least three of which are odd and at least two of which are even



(b) Draw a graph with six vertices at most three of which are odd and at least two of which are even



(All are even...)

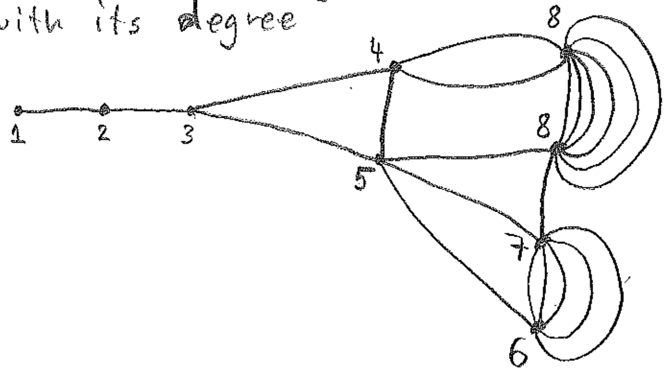
(c) Is it possible to find a graph that satisfies the conditions ~~of~~ both 4a and 4b simultaneously? Explain.

This means that we should have exactly three odd vertices but according to Corollary 9.2.6, the number of odd vertices is always even. The answer is therefore NO.

6(5): (a) Does there exist a graph with degree sequence 6, 6, 5, 5, 4, 4, 4, 3 ? Explain. The answer is NO since the the number of odd vertices is odd.

(b) Same as (a) but with 8, 8, 7, 6, 5, 4, 3, 2, 1

We can draw such a graph step by step. Each vertex is labeled with its degree



12(8): Suppose G is a graph with n vertices, n edges and no vertex of degree 0 or 1. Prove that every vertex has degree 2.

Proof: Induction over n, call the statement that theorem holds for n, S_n.

S_1: One edge, one vertex only possibility is degree 2.

S_2: Two edges, two vertices none with 0 or 1 degree, only possibilities are or

Assume that S_k is true and prove that S_{k+1} is true. Assume therefore that G is a graph with $k+1$ vertices, $k+1$ edges and no vertex of degree 0 or 1. We will first show that there exists a vertex of degree 2. If no such vertex existed, then each vertex would have degree 3 or greater, and then $\sum_{v \in V} \deg v = 2|E|$ could not hold since $|V| = |E| = k+1$. So fix v, a vertex of degree 2 in G. Either v is isolated of this type * or v is connected to two other vertices u & w via two distinct edges e_1 & e_2 $u \overset{e_1}{—} v \overset{e_2}{—} w$. Remove v and e_2 and instead let e_1 connect u & w. The resulting graph G' has k edges, k vertices and no vertex has degree 0 or 1. Since S_k is true, all vertices in G' has degree 2. Re-introducing v to reform G does not change this and so S_k holds.

In conclusion: S_1 true $\Rightarrow S_2$ true $\Rightarrow S_3$ true $\Rightarrow \ldots$ and on and on gives, according to the Induction Axiom:

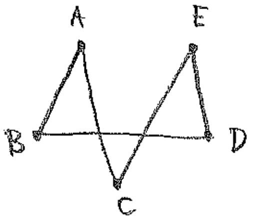
$\forall n \geq 1 : S_n$ QED.

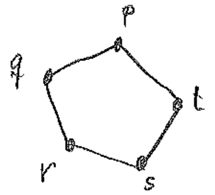
* then S_{k+1} trivially holds. Why?

15(9): For each pair of graphs shown below

- if they are not isomorphic, explain why not
- if the are isomorphic, exhibit an isomorphism from one to the other

(a)

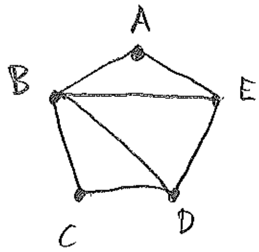


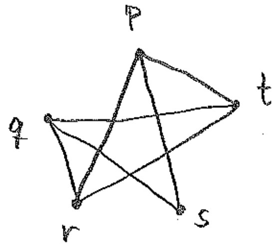


Isomorphic. An isomorphism φ is given by

$q = \varphi(A)$, $r = \varphi(B)$, $s = \varphi(D)$, $t = \varphi(E)$, and $p = \varphi(C)$

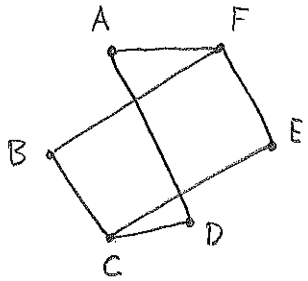
(b)

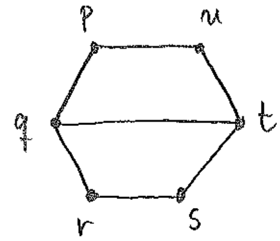




Not isomorphic since A & C are two distinct vertices of degree 2, but there is only one vertex of degree 2 among p, q, r, s, t (it is s).

(c)





Not isomorphic since F & C are two vertices of degree 3 and they are not connected with an edge, but q & t are the only vertices of degree 3 and they are connected with an edge.