It is comforting to know that we can obtain this canonical decomposition efficiently (that is, in polynomial time) via the Edmonds Matching Algorithm. But we shall postpone the algorithmic aspects of this question until Chapter 9 when we study matching algorithms.

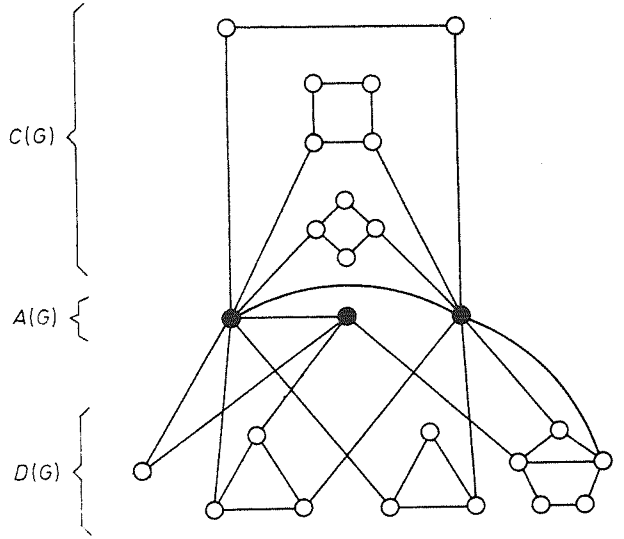

FIGURE 3.2.1. The Gallai-Edmonds decomposition of a graph G

First, let us determine the important properties of this canonical decomposition. Let G be any graph. Denote by $D(G)$ the set of all points in G which are not covered by at least one maximum matching of G. Let $A(G)$ be the set of points in $V(G)-D(G)$ adjacent to at least one point in $D(G)$. Finally let $C(G)=V(G)-A(G)-D(G)$.

The reader is invited to check the decomposition of the graph G in Figure 3.2.1. Note that $\nu(G)=12$.

A **near-perfect** matching in a graph G is one covering all but exactly one point of G.

3.2.1. THEOREM. (*The Gallai-Edmonds Structure Theorem*).
If G is a graph and $D(G)$, $A(G)$, and $C(G)$ are defined as above, then:
(a) *the components of the subgraph induced by $D(G)$ are factor-critical,*
(b) *the subgraph induced by $C(G)$ has a perfect matching,*

(c) *the bipartite graph obtained from G by deleting the points of $C(G)$ and the lines spanned by $A(G)$ and by contracting each component of $D(G)$ to a single point has positive surplus (as viewed from $A(G)$),*
(d) *if M is any maximum matching of G, it contains a near-perfect matching of each component of $D(G)$, a perfect matching of each component of $C(G)$ and matches all points of $A(G)$ with points in distinct components of $D(G)$,*
(e) $\nu(G) = \frac{1}{2}\Big(|V(G)| - c(D(G)) + |A(G)|\Big)$, *where $c(D(G))$ denotes the number of components of the graph spanned by $D(G)$.*

The statements in this theorem will follow relatively easily from part (a) of the following result which will be referred to later as the "Stability Lemma". Parts (b) and (c) will not be used until later, but it is natural to include them here.

3.2.2. LEMMA. (*The Stability Lemma*). *Let G be any graph and let $A(G)$, $C(G)$ and $D(G)$ be as defined above.*
(a) *Let $u \in A(G)$. Then $A(G-u) = A(G)-u$, $C(G-u) = C(G)$ and $D(G-u)=D(G)$.*
(b) *Let $u \in C(G)$. Then $A(G-u) \supseteq A(G)$, $C(G-u) \subseteq C(G)-u$ and $D(G-u) \supseteq D(G)$.*
(c) *Let $u \in D(G)$. Then $A(G-u) \subseteq A(G)$, $C(G-u) \supseteq C(G)$ and $D(G-u) \subseteq D(G)-u$.*

PROOF. (a) It clearly suffices to show $D(G-u)=D(G)$. Let M be a maximum matching of G. Then M covers u, since $u \notin D(G)$ (in fact, M covers $A(G)$). Hence $\nu(G-u)=\nu(G)-1$. Moreover, if M is a maximum matching of G, $M-u$ is a maximum matching of $G-u$.

First we show $D(G) \subseteq D(G-u)$. Choose any $v \in D(G)$. Let M_v be a maximum matching of G which misses v. Then M_v-u is a maximum matching of $G-u$ and moreover, M_v-u misses v too, so $v \in D(G-u)$. Thus $D(G) \subseteq D(G-u)$.

To show $D(G-u) \subseteq D(G)$, choose a point $v \in D(G-u)$. Then there is a maximum matching M' of $G-u$ which misses v. Let w be any point in $D(G)$ adjacent to u in G and let M be a maximum matching of G which misses w. If M misses v, then $v \in D(G)$ follows, so suppose that M covers v. Consider $M \cup M'$. By definition, M' avoids v. Thus the component of $M \cup M'$ covering v must be a path P starting at v with a line of $M-M'$.

Suppose P ends with a line of M'. Then the symmetric difference of M and $E(P)$ is a new matching M'' in G which misses v. Moreover, $|M''|=|M|$ so M'' is a maximum matching and $v \in D(G)$.