SF1625 Calculus in one variable
Exam
Monday January 11th 2016

Time: 08:00-13:00
No calculators, formula sheets etcetera allowed
Examiner: Lars Filipsson

This exam consists of nine problems, each worth four points, hence the maximal score is 36. Part A consists of the three first problems. To the score on part A your bonus points are added, up to a maximum of 12. The score on part A is at most 12, bonus points included. The bonus points are added automatically. You can check how many bonus points you have on your results page.

The following three problems constitute part B and the last three problems part C. You need a certain amount of points from part C to obtain the highest grades.

The grading will be performed according to this table:

Grade	A	B	C	D	E	Fx
Total score	27	24	21	18	16	15
score on part C	6	3	–	–	–	–

To obtain a maximal 4 for a solution to a problem on the exam, your solution must be well presented and easy to follow. Notation must be explained, the logical structure of the solution must be clearly described in words or in symbols and the reasoning leading up to the conclusion must be well motivated and clearly explained. Solutions that are clearly inadequate in these respects will be awarded no more than 2 points.

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PART A

1. We study the function f given by $f(x) = x - 2\arctan x$.
   A. Determine the domain of definition of f.
   B. Find the intervals where f is increasing and decreasing, respectively.
   C. Find all local extreme values of f.
   D. Find all asymptotes to the graph $y = f(x)$.
   E. Sketch using the above the graph $y = f(x)$.

2. Compute the integrals:
   A. $\displaystyle\int_0^{\ln 3} \frac{e^x}{1+e^x}\,dx$ (you may want to use the substitution $u = 1 + e^x$)
   B. $\displaystyle\int_1^2 \frac{dx}{x^2 - 3x - 4}$ (you may want to use partial fractions)

3. Find the solution to the differential equation
$$2y''(t) - 20y'(t) + 50y(t) = t$$
satisfying the initial conditions $y(0) = 1/125$ och $y'(0) = 1$.

PART B

4. Compute the integral $\displaystyle\int_0^{1/2} \frac{1}{2+8x^2}\,dx$.

   (For a maximum score the integral should be computed exactly, but an approximation may be given a partial score. Simplify answer.)

5. We study the equation $e^x + \arcsin x = 0$.
   A. For which real numbers x is the expression $e^x + \arcsin x$ defined?
   B. Show that the equation $e^x + \arcsin x = 0$ has *exactly one* solution.
   C. Find an approximate value of the solution with an error less than 0.5.

6. A part of a sphere cut off by a plane is called a *spherical cap*.
   A. Compute the volume of the spherical cap obtained by letting the area between the curve $y = \sqrt{100 - x^2}$ and the x-axis, on the interval $10 - h \leq x \leq 10$, rotate around the x-axis (we assume $0 < h < 10$).
   B. A sphere with radius 10 meters is filled with water at the rate 0.2 m^3 per minute. At what speed is the surface rising at the time when the water depth h (at the deepest spot) is 2 meters? *(Hint: from the solution of problem A you get that the volume V and the depth h satisfy $V = 10h^2\pi - h^3\pi/3$.)*

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PART C

7. This problem is about the theory of the concepts continuity and differentiability.
   A. Define what it means for a function to be continuous at a point a.
   B. Define what it means for a function to be differentiable at a point a.
   C. Prove that a function differentiable at a must be continuous at a.
   D. Give an example showing that a continuous function does not have to be differentiable.

8. We study the function f given by $f(x) = x^2 + \int_0^x \sin^2 t\, dt$.
   A. Compute the Taylor polynomial of degree 2 of f around the point $x = 0$.
   B. State the error term and show that it is bounded for $|x| \leq 1$.
   C. Compute the limit $\lim_{x \to 0} \frac{f(x)}{x^2}$.

9. Find real numbers a and b such that $a \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq b$. For a maximum score your numbers should satisfy $b - a \leq 0.2$ (and you need to provied a correct argument for this).