KTH Teknikvetenskap

# SF1625 Envariabelanalys
# Lösningsförslag till tentamen 2016-01-11

## DEL A

1. We study the function $f$ given by $f(x) = x - 2\arctan x$.

   A. Determine the domain of definition of $f$.
   B. Find the intervals where $f$ is increasing and decreasing, respectively.
   C. Find all local extreme values of $f$.
   D. Find all asymptotes to the graph $y = f(x)$.
   E. Sketch using the above the graph $y = f(x)$.

*Lösning.* A. We see that $f(x)$ is defined for all real $x$, and so the domain of definition is $\mathbf{R}$.

B , C. We get $f'(x) = 1 - \dfrac{2}{1+x^2} = \dfrac{x^2-1}{1+x^2}$ that exists for all $x$ and is $0$ iff $x = \pm 1$ Hence the critical points are $x = \pm 1$.

C. TWe study the sign of the derivative:

If $x < -1$ then $f'(x)$ is positive. It follows that $f$ is increasing.

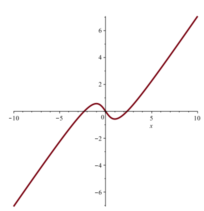
If $-1 < x < 1$ then $f'(x)$ is negative. It follows that $f$ is decreasing.

If $x > 1$ then $f'(x)$ is positive. It follows that $f$ is increasing.

We conclude that $f$ has a local max at $x = -1$ and a local minimum at $x = 1$ and no other local extreme values. The function values at these points are $f(-1) = -1 + \pi/2$ and $f(1) = 1 - \pi/2$.

D. Since $\lim_{x\to\pm\infty} \arctan x = \pm\pi/2$ we see that $f$ has an asymptote $y = x - \pi$ at infinity and an asymptote $y = x + \pi$ at minus infinity.

E. We sketch the curve:



□

**Svar:** A. All $x$. B. Increasing on $x \leq -1$. Decreasing on $-1 \leq x \leq 1$. Increasing on $x \geq 1$ . C. Local max at $x = -1$ and local min at $x = 1$. D. $y = x - \pi$ at infinity and $y = x + \pi$ at minus infinity. E. Se above.

2. Compute the integrals:

A. $\displaystyle\int_0^{\ln 3} \frac{e^x}{1+e^x}\,dx$ (you may want to use the substitution $u = 1 + e^x$)

B. $\displaystyle\int_1^2 \frac{dx}{x^2-3x-4}$ (you may want to use partial fractions)

*Lösning.* A. Using $u = 1 + e^x$, with $e^x\,dx = du$ and new interval of integration between $2$ and $4$, we get

$$\int_0^{\ln 3} \frac{e^x}{1+e^x}\,dx = \int_2^4 \frac{1}{u}\,du = [\ln u]_2^4 = \ln 2.$$

B. Since $x^2 - 3x - 4 = (x+1)(x-4)$ we get the partial fraction expansion

$$\frac{1}{x^2-3x-4} = \frac{A}{x-4} + \frac{B}{x+1}$$

where $A = 1/5$ and $B = -1/5$. We compute the integral:

$$\int_1^2 \frac{dx}{x^2-3x-4} = \int_1^2 \left(\frac{1/5}{x-4} - \frac{1/5}{x+1}\right) dx = \frac{1}{5}[\ln|x-4| - \ln|x+1|]_1^2 = \frac{2}{5}\ln\frac{2}{3}.$$

□

**Svar:** A. $\ln 2$

B. $\frac{2}{5}\ln\frac{2}{3}$

3. Find the solution to the differential equation
$$2y''(t) - 20y'(t) + 50y(t) = t$$
satisfying the initial conditions $y(0) = 1/125$ och $y'(0) = 1$.

*Lösning.* The solution $y$ has the structure $y = y_h + y_p$ where $y_h$ is the complete solution to the corresponding homogeneous equation and $y_p$ is some particular solution.

First we find $y_h$. The characteristic equation $2r^2 - 20r + 50 = 0$ has the solution $r = 5$, and so we get
$$y_h(t) = (A + Bt)e^{5t}.$$
Since the right hand side is a first degree polynomial we guess $y_p(t) = ct + d$. The derivative of this is c and the second derivative is zero and when we plug that into the differential equation we obtain $-20c + 50(ct + d) = t$, hence $c = 1/50$ and $d = 1/125$.

The solution to the differential equation is therefore
$$y(t) = (A + Bt)e^{5t} + \frac{t}{50} + \frac{1}{125}.$$
The condition $y(0) = 1/125$ yields $A = 0$. The condition $y'(0) = 1$ then yields $B = 49/50$.

The solution to differential equation satisfying the conditions is therefore
$$y(t) = \frac{49t}{50}e^{5t} + \frac{t}{50} + \frac{1}{125}.$$
□

**Svar:** $y(t) = \frac{49t}{50}e^{5t} + \frac{t}{50} + \frac{1}{125}$.

## DEL B

4. Compute the integral $\int_0^{1/2} \frac{1}{2+8x^2}\,dx$.

*(For a maximum score the integral should be computed exactly, but an approximation may be given a partial score. Simplify answer. )*

*Lösning.* We observe that:

$$\frac{1}{2+8x^2} = \frac{1}{2} \cdot \frac{1}{1+(2x)^2}.$$

Now we can compute the integral:

$$\int_0^{1/2} \frac{1}{2+8x^2}\,dx = \frac{1}{2}\int_0^{1/2} \frac{1}{1+(2x)^2}\,dx = \frac{1}{4}[\arctan(2x)]_0^{1/2} = \frac{\pi}{16}.$$

□

**Svar:** $\pi/16$

5. We study the equation $e^x + \arcsin x = 0$.
   A. For which real numbers $x$ is the expression $e^x + \arcsin x$ defined?
   B. Show that the equation $e^x + \arcsin x = 0$ has *exactly one* solution.
   C. Find an approximate value of the solution with an error less than $0.5$.

*Lösning.* The expression is defined for $-1 \leq x \leq 1$. Put $f(x) = e^x + \arcsin x$. The equation can then be written $f(x) = 0$. The domain of definition of $f$ is $-1 \leq x \leq 1$ and $f$ is continuous on the closed and bounded interval. Since $f(-1) = e^{-1} - \pi/2 < 0$ and $f(1) = e^1 + \pi/2 > 0$ the intermediate value theorem gurarantees a point $x^*$ between $-1$ and $1$ such that $f(x^*) = 0$. In fact $x^*$ must lie between $-1$ and $0$ since $f(0) = 1 > 0$ . An approximation of $x^*$ with an error less than $0.5$ is therefore $-0.5$.

Since $f'(x) = e^x + \dfrac{1}{\sqrt{1-x^2}}$ is positive on the open interval, $f$ is increasing. Therefore there cannot be more than one solution to the equation.

□

**Svar:** See the solution. Approximate value $-0.5$

6. A part of a sphere cut off by a plane is called a *spherical cap*.
   A. Compute the volume of the spherical cap obtained by letting the area between the curve $y = \sqrt{100 - x^2}$ and the $x$-axis, on the interval $10 - h \leq x \leq 10$, rotate around the $x$-axis (we assume $0 < h < 10$).
   B. A sphere with radius $10$ meters is filled with water at the rate $0.2$ m$^3$ per minute. At what speed is the surface rising at the time when the water depth $h$ (at the deepest spot) is 2 meters? *(Hint: from the solution of problem A you get that the volume $V$ and the depth $h$ satisfy $V = 10h^2\pi - h^3\pi/3$.)*

*Lösning.* A. According to the formula for volumes of revolution the volume $V$ is given by

$$V = \pi \int_{10-h}^{10} (100 - x^2)\, dx = 10h^2\pi - \frac{h^3\pi}{3}.$$

B. We use the answer to A and get that the volume $V$ when the depth is $h$ is given by

$$V = 10h^2\pi - \frac{h^3\pi}{3}$$

where both $V$ and $h$ depend on $t$. We differentiate with respect to $t$ and get

$$\frac{dV}{dt} = 20h\pi\frac{dh}{dt} - h^2\pi\frac{dh}{dt}.$$

We know the the volume changes by $0.2$ m3 per minute, and so $dV/dt = 0.2$. We need to find $dh/dt$ when $h = 2$. When $h = 2$ we see that

$$0.2 = 40\pi\frac{dh}{dt} - 4\pi\frac{dh}{dt},$$

and so

$$\frac{dh}{dt} = \frac{1}{180\pi} \text{ meters per minute.}$$

□

**Svar:** A.$10h^2\pi - \frac{h^3\pi}{3}$
B. $1/180\pi$ meters per minute

## DEL C

7. This problem is about the theory of the concepts continuity and differentiability.
   A. Define what it means for a function to be continuous at a point $a$.
   B. Define what it means for a function to be differentiable at a point $a$.
   C. Prove that a function differentiable at $a$ must be continuous at $a$.
   D. Give an example showing that a continuous function does not have to be differentiable.

*Lösning.* See the text book. □

**Svar:** See the text book.

8. We study the function $f$ given by $f(x) = x^2 + \int_0^x \sin^2 t\, dt$.

   A. Compute the Taylor polynomial of degree $2$ of $f$ around the point $x = 0$.
   B. State the error term and show that it is bounded for $|x| \leq 1$.
   C. Compute the limit $\lim_{x\to 0} \frac{f(x)}{x^2}$.

*Lösning.* Using the fundamental theorem and the chain rule we get
$f'(x) = 2x + \sin^2 x$ and $f'(0) = 0$,
$f''(x) = 2 + 2\sin x \cos x$ and $f''(0) = 2$,
$f'''(x) = 2\cos 2x$.
It is clear that $f(0) = 0$ and so the Taylor polynomial is $p(x) = x^2$.

The error term is $\frac{2\cos 2c}{3!}x^3$ for some $c$ between $0$ and $x$. Since $|\cos 2c| \leq 1$ the absolute value of the error is $\leq 2/3! = 1/3$ when $|x| \leq 1$. The error term is therefore bounded.

Using the above we get

$$\lim_{x\to 0} \frac{f(x)}{x^2} = \lim_{x\to 0} \frac{x^2 + \frac{2\cos 2c}{3!}x^3}{x^2} = 1.$$

□

**Svar:** A. $p(x) = x^2$. B. See the solution. C. $1$

9. Find real numbers $a$ and $b$ such that $a \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq b$. For a maximum score your numbers should satisfy $b - a \leq 0.2$ (and you need to provied a correct argument for this).

*Lösning.* Clearly $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \sum_{n=3}^{\infty} \frac{1}{n^2}$.

Using estimates like those in the proof of the integral criterion it is easy to see that

$$\int_3^{\infty} \frac{1}{x^2}\, dx < \sum_{n=3}^{\infty} \frac{1}{n^2} < \frac{1}{9} + \int_3^{\infty} \frac{1}{x^2}\, dx.$$

Putting this together and computint the integral (its value is $1/3$), we get

$$1 + \frac{1}{4} + \frac{1}{3} < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \frac{1}{4} + \frac{4}{9}.$$

Since $1 + 1/4 + 1/3 = 19/12 = 1.58\dots$ and $1 + 1/4 + 4/9 = 61/36 = 1.69\dots$ we can choose $a = 1.58$ and $b = 1.7$. Hence

$$1.58 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1.7.$$

□

**Svar:** Se lösningen.

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