KTH Teknikvetenskap

**SF1624 Algebra och geometri**
**Exam**
**Wednesday, January 13, 2016**

Time: 08:00am–1:00pm
No books/notes/calculators etc. allowed
Examiner: Tilman Bauer

This exam consists of nine problems, each worth 4 points.

Part A comprises the first three problems. The bonus points from the seminars will be automatically added to the total score of this part, which however cannot exceed 12 points.

The next three problems constitute part B, and the last three problems part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

| Grade | A | B | C | D | E | Fx |
|---|---|---|---|---|---|---|
| Total sum | 27 | 24 | 21 | 18 | 16 | 15 |
| of which in part C | 6 | 3 | – | – | – | – |

To get full score on a problem, your solution must be well-presented and easy to follow. In particular, you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument. Solutions severely lacking in these respects will achieve at most 2 points.

*p.t.o.*

## PART A

1. Let $A = (1, -1, 1)$, $B = (1, 3, 1)$, $C = (1, 1, 0)$ be points in $\mathbb{R}^3$.
   (a) Describe the plane $P$ containing $A$, $B$ och $C$ in parametric form and give a system of linear equations which describes $P$. **(2 p)**
   (b) Let $L$ be the line through $A$ and $B$. Compute the distance between $C$ and the line $L$. **(2 p)**

2. Consider the following matrix:
$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$
   (a) Determine if the vector $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ lies in the image $\mathrm{im}(A)$. **(2 p)**
   (b) Determine a basis for the null space $\ker(A)$. **(2 p)**

3. Let
$$A = \begin{bmatrix} -1 & 4 \\ 0 & 1 \end{bmatrix}.$$
   (a) Compute the eigenvalues and corresponding eigenvectors for the matrix $A$. **(1 p)**
   (b) Find a $2 \times 2$-matrix $S$ such that $S^{-1}AS$ is a diagonal matrix. **(1 p)**
   (c) Compute $A^{139}$. **(2 p)**

## Part B

4. To find the coefficient of thermal expansion $\lambda$ for a metal experimentally, a metal rod was heated up and its length was measured. Apply the least-squares method to find $\lambda$ from this data.

| Temp ($C^{\circ}$) | 20 | 22 | 24 | 26 |
|---|---|---|---|---|
| Längd (mm) | 1 | 2 | 4 | 5 |

The following linear relation between the temperature $T$ and the length $L$ holds:
$$L(T) = L_0 + L_1(T - T_m),$$
where $T_m = 23$ is the average of the four temperatures. The coefficient of thermal expansion $\lambda$ can be computed from the relation $L_1 = \lambda L_0$.

5. (a) Explain why there exists one and only one linear map $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ such that
$$f\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}2\\1\\-1\end{bmatrix}, \quad f\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\-1\\-1\end{bmatrix} \quad \text{and} \quad f\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-1\\-3\end{bmatrix}.$$
**(2 p)**

   (b) Find the matrix for $f$ with respect to the standard bases. **(2 p)**

6. The vector space $W$ is spanned by the basis $\mathcal{B} = \{\vec{u}, \vec{v}\}$, where
$$\vec{u} = \begin{bmatrix}-1\\-1\\-1\end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix}2\\0\\1\end{bmatrix}.$$

   (a) Let $\mathcal{C}$ be another basis for $W$ such that the matrix $T = \begin{bmatrix}1 & 1\\2 & 3\end{bmatrix}$ is the change-of-basis matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$. Find $\mathcal{C}$. **(2 p)**

   (b) Determine if the vector $\begin{bmatrix}0\\2\\1\end{bmatrix}$ lies in $W$. If it does, compute its coordinates in the bases $\mathcal{B}$ and $\mathcal{C}$. **(2 p)**

*p.t.o.*

## PART C

7. For an $n \times n$-matrix
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$
the sum $a_{11} + a_{22} + \cdots + a_{nn}$ of the diagonal elements is called the **trace** of $A$ and is denoted by $\mathrm{tr}(A)$.
   (a) Let $A$ and $B$ be $n \times n$-matrices. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Conclude that $\mathrm{tr}(A) = \mathrm{tr}(B^{-1}AB)$, assuming that $B$ is invertible. **(2 p)**
   (b) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Let $M$ be the matrix for $f$ with respect to a basis $\mathcal{B}$. The trace of the map $f$ is defined to be the trace of $M$. Show this is well-defined, i. e. that the trace of $f$ is independent of the choice of basis. **(2 p)**

8. Let $A$ be a symmetric, invertible matrix.
   (a) Show that the inverse $A^{-1}$ is also symmetric. **(2 p)**
   (b) Show that $(\vec{x})^T A\vec{x}$ is a positive definite quadratic form if and only if $(\vec{x})^T A^{-1}\vec{x}$ is a positive definite quadratic form. **(2 p)**

9. Let $\vec{v}_1$, $\vec{v}_2$, och $\vec{v}_3$ be orthonormal vectors in $\mathbb{R}^3$. Compute the absolute value of the determinant
$$\left|\det\left[\vec{v}_1 + \vec{v}_2 \quad \vec{v}_2 + \vec{v}_3 \quad \vec{v}_3 + \vec{v}_1\right]\right|.$$
**(4 p)**