Lecture 1: Channel Equalization 1
Advanced Digital Communications (EQ2410)[1]

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13:00-15:00, E34

[1] Textbook: U. Madhow, *Fundamentals of Digital Communications*, 2008

Notes

Overview

Notes

Channel Model

Intersymbol interference (ISI)

$\Rightarrow$ Successive symbols interfere with each other.

ISI is caused by

- Multi-path propagation
  - Radio communications: signals are reflected by walls, buildings, hills, ionosphere, ...
  - Underwater communications: signals are reflected by the ground, the surface, interface between different water layers,...
- Frequency-selective and bandlimited channels
  - Cables and wires are modeled by (linear) LCR circuits.
  - Frequency division multiplexing (FDM) requires limited bandwidth per channel.

Mathematical model

- ISI can be modeled by a linear filter.
  (implicit assumption: linearity)
- In general: *time-variant* linear filter.

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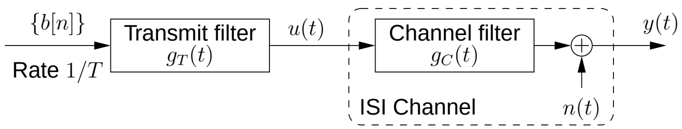
Channel Model

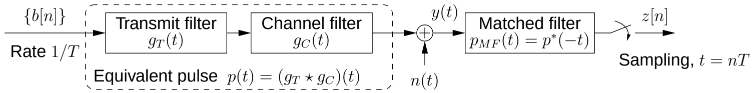


- Transmitted signal: $u(t) = \sum_{n=-\infty}^{\infty} b[n] g_T(t - nT)$
  - $\{b[n]\}$: symbol sequence transmitted at rate $1/T$
  - $g_T(t)$: impulse response of the transmit filter
  - T: duration of one symbol
- Received signal: $y(t) = \sum_{n=-\infty}^{\infty} b[n] p(t - nT) + n(t)$
  - $p(t) = (g_T \star g_C)(t)$: impulse response of the cascade of the transmit and channel filters[2].
  - $g_C(t)$: channel impulse response
  - $n(t)$: complex additive white Gaussian noise (AWGN) with variance $\sigma^2 = N_0/2$ per dimension
- Channel equalization: extract $\{b[n]\}$ from $y(t)$

[2]Convolution of two signals $a(t)$ and $b(t)$: $q(t) = (a \star b)(t) = \int a(u) b(t-u) du$.

Notes

Receiver Front End



Theorem (Optimality of the Matched Filter)

The optimal receiver filter is matched to the equivalent pulse $p(t)$ and is specified in the time and frequency domain as follows:

$$g_{R,opt}(t) = p_{MF}(t) = p^*(-t)$$
$$G_{R,opt}(f) = P_{MF}(f) = P^*(f).$$

In terms of a decision on the symbol sequence $\{b[n]\}$, there is no loss of relevant information by restricting attention to symbol rate samples of the matched filter output given by

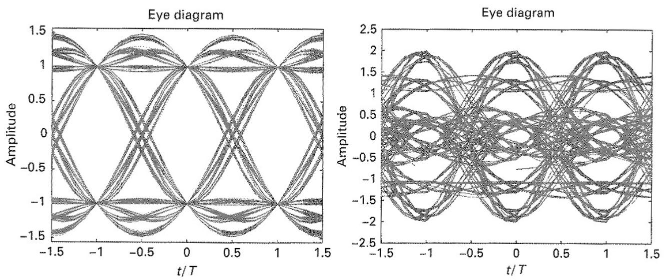
$$z[n] = (y \star p_{MF})(nT) = \int y(t)p_{MF}(nT - t)dt = \int y(t)p^*(t - nT)dt.$$

[U. Madhow, Fundamentals of Dig. Comm., 2008]

Notes

Eye Diagrams

- Visualization of the effect of ISI (for the noise-free case)
- Received signal (noise free): $r(t) = \sum_n b[n]x(t - nT)$
- Effective impulse response: $x(t) = (g_T \star g_C \star g_R)(t)$ (incl. transmit, channel, and receive filter)
- Eye diagram
  → superimpose the waveforms $\{r(t - kT),\ k = \pm 1, \pm 2, \ldots\}$
- Example
  (a) BPSK signal with ISI free pulse in (open eye);
  (b) BPSK signal with ISI (closed eye).

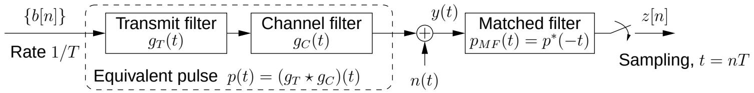


(a) Open eye (b) Closed eye

[U. Madhow, *Fundamentals of Dig. Comm.*, 2008]

Notes

Nyquist Criterion



Theorem (Nyquist[3] Criterion and Nyquist Rate)

The received signal after sampling (sampling rate $1/T$) is given as

$$z(nT) = \sum_{m=\infty}^{\infty} b[m] \cdot x(nT - mT) + n(nT) = b[n] \cdot x(0) + \sum_{m \neq n} b[m] \cdot x(nT - mT) + n(nT),$$

with the effective impulse response: $x(t) = (g_T \star g_C \star g_R)(t)$. Under the assumption that $X(f) = \mathcal{F}\{x(t)\} = G_T(f)G_C(f)G_R(f) = 0$ for $|f| > W$, the transmission system is ISI free if

$$x(nT) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases} \quad \Leftrightarrow \quad \sum_m X\left(f - \frac{m}{T}\right) = T.$$

ISI-free transmission at symbol rate R is possible if $0 < R \leq R_N$ where the upper bound R_N is given by the Nyquist rate $R_N = 2W$.

[3] Harry Nyquist 1928 (Swedish/American inventor)

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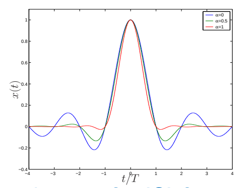
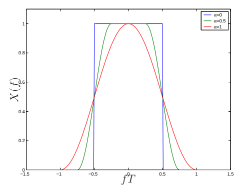
Nyquist Criterion

(a)



(b)



Pulse shaping for ISI-free transmission

- Raised-cosine pulses can be designed to be ISI free for $0 < R \leq 2W$
- In time domain (see plot (a))

$$x_{rc}(t) = \text{sinc}\left(\frac{t}{T}\right) \frac{\cos \pi\alpha t/T}{1 - 4\alpha^2 t^2/T^2}$$

- In frequency domain (see plot(b))

$$X_{rc}(f) = \begin{cases} T & , \text{ for } |f| \leq \frac{1-\alpha}{2T} \\ \frac{T}{2}\left[1 + \cos\left(\frac{\pi T}{\alpha}\left(|f| - \frac{1-\alpha}{2T}\right)\right)\right] & , \text{ for } \frac{1-\alpha}{2T} < |f| < \frac{1+\alpha}{2T} \\ 0 & , \text{ for } |f| > \frac{1+\alpha}{2T} \end{cases}$$

- Design of the transmit and receive filters (matched filters):

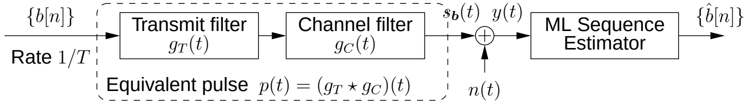
$$|G_T(f)| = K_1 \frac{|X_{rc}(f)|^{1/2}}{|G_C(f)|^{1/2}} \quad \text{and} \quad |G_R(f)| = K_2 \frac{|X_{rc}(f)|^{1/2}}{|G_C(f)|^{1/2}}$$

with K_1 so that $\int_{-\infty}^{\infty} g_T^2(t)dt = E_b$ and K_2 arbitrary.

Notes

Maximum Likelihood Sequence Estimation

Based on the continuous-time model



- Goal: find $\mathbf{b}$ that maximizes the likelihood function[4]

$$L(y|\mathbf{b}) = \frac{p(y|\mathbf{b})}{p(y)} = \exp\left(\frac{1}{\sigma^2}(\mathrm{Re}(\langle y, s_{\mathbf{b}}\rangle) - ||s_{\mathbf{b}}||^2/2)\right) \quad \text{with} \quad s_{\mathbf{b}}(t) = \sum_n b[n]p(t-nT).$$

- Or equivalently: find $\mathbf{b}$ that maximizes the cost function

$$\Lambda(\mathbf{b}) = \mathrm{Re}(\langle y, s_{\mathbf{b}}\rangle) - ||s_{\mathbf{b}}||^2/2$$

- Brute-force detector
  - try out all realizations of $\mathbf{b}$
  - $\Rightarrow$ not feasible: N symbols with M-ary modulation lead to M^N possible sequences $\mathbf{b}$.

[4] Inner product of two signals $a(t)$ and $b(t)$: $\langle a, b\rangle = \int a(t)b^*(t)dt$.

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Maximum Likelihood Sequence Estimation
– Decomposition of $\Lambda(\mathbf{b})$

- Useful definition:

$$h[m] = \int p(t)p^*(t-mT)dt = (p \star p_{MF})(mT) = x(mT)$$

$\rightarrow$ sampled effective impulse response (transmit/channel/receiver filter)
$\rightarrow$ useful property: $h[-m] = h^*[m]$

- First term in $\Lambda(\mathbf{b})$ (see e.g. textbook, p. 203)

$$\mathrm{Re}(\langle y, s_{\mathbf{b}}\rangle) = \mathrm{Re}\left(\sum_n b^*[n]\int y(t)p^*(t-nT)dt\right) = \sum_n \mathrm{Re}(b^*[n]z[n])$$

- Second term in $\Lambda(\mathbf{b})$ (see e.g. textbook, p. 206)

$$\begin{aligned} ||s_{\mathbf{b}}||^2 &= \langle s_{\mathbf{b}}, s_{\mathbf{b}}\rangle = \sum_n\sum_m b[n]b^*[m]h[m-n] = \dots \\ &= h(0)\sum_n |b[n]|^2 + \sum_n\sum_{m<n} 2\mathrm{Re}(b^*[n]b[m]h[n-m]) \end{aligned}$$

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Maximum Likelihood Sequence Estimation
– Decomposition of $\Lambda(\mathbf{b})$

Intermediate result

$$\Lambda(\mathbf{b}) = \sum_n \left\{ \mathrm{Re}(b^*[n]z[n]) - \frac{h[0]}{2}|b[n]|^2 - \mathrm{Re}\left(b^*[n] \sum_{m<n} b[m]h[n-m] \right) \right\}$$

→ The cost function is additive in n.

→ The n-th summand of the sum is a function of the "current" symbol $b[n]$ and the "past" symbols $\{b[m], m < n\}$.

→ Interpretation: the sum over m removes the ISI from previously transmitted symbols from $z[n]$.

Notes

Maximum Likelihood Sequence Estimation
– Viterbi Algorithm

- Assumption: the system has a limited impulse response, i.e., $h[n] = 0$, $|n| > L$, and we get

$$\Lambda(\mathbf{b}) = \sum_n \{ \mathrm{Re}(b^*[n]z[n]) - \frac{h[0]}{2}|b[n]|^2 - \mathrm{Re}\left(b^*[n] \sum_{m=n-L}^{n-1} b[m]h[n-m] \right) \}$$

  → good approximation for practical systems!
  → only the previous L symbols cause ISI.
- State definition: $s[n] = (b[n-L], \ldots, b[n-1])$, M^L states.
- Branch metric:

$$\lambda_n(s[n] \to s[n+1]) = \lambda_n(b[n], s[n])$$
$$= \mathrm{Re}(b^*[n]z[n]) - \frac{h[0]}{2}|b[n]|^2 - \mathrm{Re}\left(b^*[n] \sum_{m=n-L}^{n-1} b[m]h[n-m] \right)$$

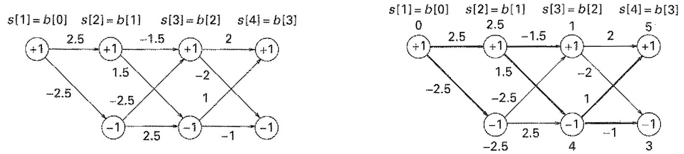
- Accumulated metric (AM) at time k

$$\Lambda_k(\mathbf{b}) = \sum_{n=1}^{k} \lambda_n(s[n] \to s[n+1]) = \lambda_k(s[k] \to s[k+1]) + \Lambda_{k-1}(\mathbf{b})$$
$$= \sum_{n=1}^{k} \lambda_n(b[n], s[n]) = \lambda_k(b[k], s[k]) + \Lambda_{k-1}(\mathbf{b})$$

Notes

Maximum Likelihood Sequence Estimation
– Example Viterbi Algorithm[5]

- BPSK-modulated signal, $h[0] = 3/2$, $h[1] = h[-1] = -1/2$, i.e., $L = 1$.



[U. Madhow, *Fundamentals of Dig. Comm.*, 2008]

→ Each length-k path through the trellis is associated with a length-k symbol sequence and an AM $\Lambda_k(\mathbf{b})$

→ At any given state, only the incoming path with the best AM $\Lambda_k(\mathbf{b})$ (survivor) has to be kept.

→ Let $\Lambda^*(1:k, s')$ be the AM of the survivor at state $s[k] = s'$. The AM for the path emerging from $s[k] = s'$ and ending at $s[k+1] = s$ is given as

$$\Lambda_0(1:k+1, s' \to s) = \Lambda^*(1:k, s') + \lambda_{k+1}(s' \to s)$$

and we have

$$\Lambda^*(1:k+1, s) = \max_{s'} \Lambda_0(1:k+1, s' \to s)$$

→ If the end of the trellis is reached, the best survivor is the maximum likelihood sequence.

[5] See Figure 5.5-6 in [U. Madhow, *Fundamentals of Dig. Comm.*, 2008]

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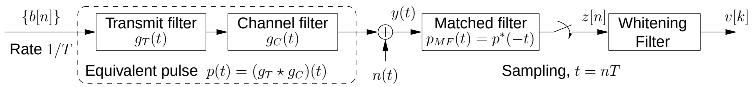
Maximum Likelihood Sequence Estimation
– Alternative Formulation

Based on the discrete-time model



- After matched filtering, the additive noise in $z[n]$ is colored; a whitening filter is required.
- Model for the received sequence:

$$v[k] = \sum_{n=0}^{L} f[n] b[k-n] + \eta_k, \qquad \text{with}$$

  → discrete-time impulse response $f[n]$ describing the cascade of transmit, channel, receive, and whitening filter;
  → complex additive white Gaussian noise η_k with noise variance σ^2 per dimension.

- Cost function to be minimized

$$g(\mathbf{b}) = \sum_k \left| v[k] - \sum_{n=0}^{L} f[n] b[k-n] \right|^2$$

→ ML sequence can be found with the Viterbi algorithm.

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