Lecture 2
Channel Equalization

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Lecture 2: Channel Equalization 2
Advanced Digital Communications (EQ2410)[1]

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10:00-12:00, B24

[1] Textbook: U. Madhow, *Fundamentals of Digital Communications*, 2008

Notes

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Overview

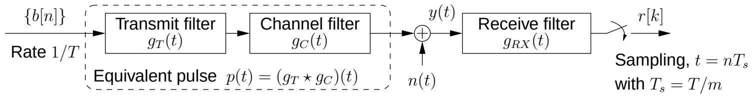
Lecture 1

- Channel model and ISI
- Optimal receiver design
- Nyquist criterion, Nyquist rate, and pulse shaping
- ML sequence estimation (optimal but high complexity)

Lecture 2: Linear Equalization (suboptimal but low(er) complexity)

Notes

Geometric Model



- Received signal: $y(t) = \sum_n b[n]p(t - nT) + n(t)$
- Arbitrary receive filter $g_{RX}(t)$
  - suboptimal equalizers do not require an optimal front end
  - not matched to $p(t)$; often wideband filter, especially, if sampling is faster than the symbol rate ($m > 1$).
- Sampling time: $T_s = T/m$
  - $m = 1$: symbol spaced sampling
  - $m > 1$: fractionally spaced sampling
- Output of the sampler: $r[k] = (y \star g_{RX})(kT_s + \delta)$
- Sampled impulse response: $f[k] = (p \star g_{RX})(kT_s + \delta)$
- Noise at the output of the sampler: $w[k] = (n \star g_{RX})(kT_s + \delta)$
  → Colored noise with auto correlation/covariance

$$C_w(k) = 2\sigma^2 \int g_{RX}(t) g^*_{RX}(t - kT_s) dt$$

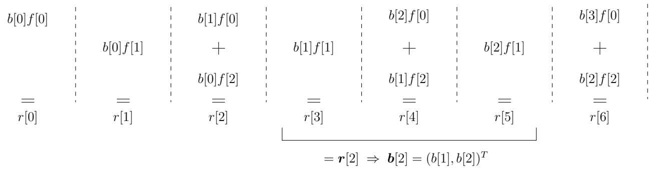
Notes

Geometric Model

- A block of L received samples $\mathbf{r}[n]$ is used to decide one symbol $b[n]$.
- Model for the received vector

$$\mathbf{r}[n] = \mathbf{U}\mathbf{b}[n] + \mathbf{w}[n],$$

where
  - $\mathbf{b}[n] = (b[n-k_1], \ldots, b[n], \ldots, b[n+k_2])^T$ is a length-K vector with $K = k_1 + k_2 + 1$; it includes all symbols $b[n']$ which contribute to $\mathbf{r}[n]$.
  - $\mathbf{U} = [\mathbf{u}_{-k_1} \ldots \mathbf{u}_{k_2}]$ is a $L \times K$ matrix with columns $\mathbf{u_i}$ corresponding to the shifted impulse response $f[k]$.
  - $\mathbf{w}[n]$ is zero-mean, proper complex Gaussian noise with covariance matrix $\mathbf{C}_w$
- Example[2]: $L = 3,\ k_1 = 1,\ k_2 = 0,\ K = 2$



$$\mathbf{r}[2] = \begin{bmatrix} r[3] \\ r[4] \\ r[5] \end{bmatrix} = \begin{bmatrix} f[1] & 0 \\ f[2] & f[0] \\ 0 & f[1] \end{bmatrix} \begin{bmatrix} b[1] \\ b[2] \end{bmatrix} + \begin{bmatrix} w[3] \\ w[4] \\ w[5] \end{bmatrix}$$

[2] Note that this choice of parameters may not be the optimal choice.

Notes

Linear Equalization

- Alternative formulation of the received vector

$$\mathbf{r}[n] = b[n]\mathbf{u}_0 \; + \sum_{i=-k_1, i\neq 0}^{k_2} b[n+i]\mathbf{u}_i \; + \; \mathbf{w}[n]$$

- Linear equalization: correlate $\mathbf{r}[n]$ with a vector $\mathbf{c}$ to produce a decision variable $Z[n] = \langle \mathbf{r}[n], \mathbf{c} \rangle$:

$$Z[n] = \mathbf{c}^H \mathbf{r}[n] = b[n]\mathbf{c}^H\mathbf{u}_0 \; + \underbrace{\sum_{i=-k_1, i\neq 0}^{k_2} b[n+i]\mathbf{c}^H\mathbf{u}_i}_{\text{residual ISI}} \; + \mathbf{c}^H\mathbf{w}[n]$$

→ choose $\mathbf{c}$ such that the signal component $b[n]\mathbf{c}^H\mathbf{u}_0$ is significantly larger than the residual ISI (if possible).

→ Colored noise $\mathbf{c}^H\mathbf{w}[n]$ with covariance $\mathbf{c}^H\mathbf{C}_w\mathbf{c}$

- Hard decision or soft decision based on $Z[n]$

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Zero-Forcing Equalization

- The zero-forcing solution (if it exists) satisfies

$$\mathbf{c}^H\mathbf{u}_0 = 1 \quad \text{and} \quad \mathbf{c}^H\mathbf{u}_i = 0, \quad \text{for all } i \neq 0$$

or equivalently

$$\mathbf{c}^H\mathbf{U} = (0, \ldots 0, 1, 0 \ldots, 0)^T = \mathbf{e}^T \quad \text{or} \quad \mathbf{U}^H\mathbf{c} = \mathbf{e}$$

- The correlator $\mathbf{c}$ should lie in the signal space spanned by the vectors $\{\mathbf{u}_i\}$
  - Components of $\mathbf{c}$ outside the signal space contribute only noise.
  - → $\mathbf{c}$ should be a linear combination of the $\mathbf{u}_i$'s, i.e., $\mathbf{c} = \mathbf{U}\mathbf{a}$.
- It follows the zero-forcing equalizer as $\mathbf{c}_{ZF} = \mathbf{U}(\mathbf{U}^H\mathbf{U})^{-1}\mathbf{e}$

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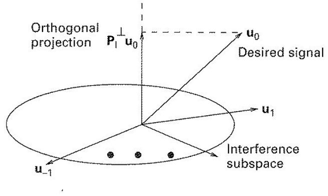
Zero-Forcing Equalization

- Geometric interpretation of the signal space
  - Component in direction of the desired signal: $\mathbf{u}_0$
  - Interference subspace $\mathbf{S}_I$ spanned by $\{\mathbf{u}_i, i \neq 0\}$



[U. Madhow, *Fundamentals of Dig. Comm.*, 2008]

- To suppress the interference completely, $\mathbf{c}$ must be orthogonal to the interference subspace (only possible if $L > K - 1$)): $\mathbf{c} = \alpha \mathbf{P}_I^{\perp} \mathbf{u_0}$
- With the normalization $\langle \mathbf{c}, \mathbf{u}_0 \rangle = 1$, we get the scale factor $\alpha = 1/\|\mathbf{P}_I^{\perp}\mathbf{u_0}\|^2$.
- Noise variance v_{ZF}^2 of the output noise:

$$v_{ZF}^2 = \sigma^2 \|\mathbf{c}\|^2 = \frac{\sigma^2}{\|\mathbf{P}_I^{\perp}\mathbf{u_0}\|^2}$$

$\rightarrow$ Noise enhancement with increasing α

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MMSE Equalization

- Minimization of the minimum mean squared error (MMSE)

$$\mathbf{c}_{MMSE} = \arg\min_{\mathbf{c}} \mathrm{E}[\underbrace{|\mathbf{c}^H\mathbf{r}[n] - b[n]|^2}_{\text{mean squared error (MSE)}}]$$

  $\rightarrow$ tradeoff between between ISI suppression and noise enhancement

- Solution:

$$\mathbf{c}_{MMSE} = \mathbf{R}^{-1}\mathbf{p}$$

  with
  - autocorrelation matrix of the received signal $\mathbf{R} = \mathrm{E}[\mathbf{r}[n]\mathbf{r}^H[n]]$
  - cross-correlation vector $\mathbf{p} = \mathrm{E}[b^*[n]\mathbf{r}[n]]$
- Proof: by solving $\nabla_{\mathbf{c}^*} \mathrm{E}[|\mathbf{c}^H\mathbf{r}[n] - b[n]|^2] = 0$
- Useful performance measure for linear equalizers: signal-to-interference ratio (SIR)

$$\mathrm{SIR} = \frac{\sigma_b^2 |\langle \mathbf{c}, \mathbf{u}_0 \rangle|^2}{\sigma_b^2 \sum_{j\neq 0} |\langle \mathbf{c}, \mathbf{u}_j \rangle|^2 + \mathbf{c}^H \mathbf{C}_w \mathbf{c}}$$

- Important properties:
  - The MMSE equalizer maximizes the SIR.
  - At high SNR, the MMSE equalizer specializes to the ZF equalizer.

Notes

Adaptive Implementations

- Channel knowledge required for equalization
  - → ZF: $\mathbf{U}$ ⇒ channel impulse response required
  - → MMSE: $\mathbf{R}, \mathbf{p}$ ⇒ auto and cross correlation of the received vector and the data
- Training and decision directed modes
  - training phase: training sequences (known to the receiver) are transmitted; the receiver estimates parameters of the EQ.
  - data phase: data are transmitted, and the receiver uses the EQ based on the estimated parameters.
- Least squares algorithm
  - MMSE with empirical averages of $\mathbf{R}$ and $\mathbf{p}$ (N training sequences)
  - → $\mathbf{c}_{LS} = \hat{\mathbf{R}}^{-1}\hat{\mathbf{p}}$ with

$$\hat{\mathbf{R}} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{r}[n]\mathbf{r}^H[n] \quad \text{and} \quad \hat{\mathbf{p}} = \frac{1}{N}\sum_{n=1}^{N} b^*[n]\mathbf{r}[n]$$

- Least mean squares algorithm
  - Gradient descent: $\mathbf{c}[k] = \mathbf{c}[k-1] - \mu \underbrace{\mathrm{E}[\mathbf{r}[n](\mathbf{r}^H[n]\mathbf{c}[k-1] - b^*[n])]}_{\nabla_{\mathbf{c}^*} J(\mathbf{c}[k-1])}$
  - Replacing the expectation by the instantaneous value yields

$$\begin{aligned}\mathbf{c}[k] &= \mathbf{c}[k-1] - \mu \mathbf{r}[n]\underbrace{(\mathbf{r}^H[n]\mathbf{c}[k-1] - b^*[n])}_{e^*[k]} \\ &= \mathbf{c}[k-1] + \mu e^*[k]\mathbf{r}[n]\end{aligned}$$

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Performance Analysis

- The equalizer output can be written as

$$Z[n] = A_0 b[n] + \sum_{i\neq 0} A_i b[n+i] + W[n] \quad \text{with}$$

  - amplitudes $A_i = \langle \mathbf{c}, \mathbf{u_i} \rangle$
  - zero-mean Gaussian noise with variance $v^2 = \sigma^2 \|\mathbf{c}\|^2$
- BPSK system with hard decision based on $Z[n]$: $\hat{b}[n] = \mathrm{sign}(Z[n])$
- Error probability

$$P_e = \Pr[\hat{b}[n] \neq b[n]] = \Pr[Z[n] > 0 | b[n] = -1]$$

  due to symmetry.
- Conditional error probability for given ISI bits $\mathbf{b}_I = \{b_{n+i}, i \neq 0\}$

$$\begin{aligned}&P_{e|\mathbf{b}_I} = \Pr[Z[n] > 0 | b[n] = -1, \mathbf{b}_I] \\ &= \Pr[W[n] > A_0 - \sum_{i\neq 0} A_i b[n+i])] = Q\left(\frac{A_0 - \sum_{i\neq 0} A_i b[n+i]}{v}\right)\end{aligned}$$

- Error probability averaged over ISI: $P_e = \mathrm{E}_{\mathbf{b}_I}\{P_{e|\mathbf{b}_I}\}$
- Alternative approximation: model ISI as Gaussian random variable with variance $v_I^2 = \sum_{i\neq 0} A_i^2$

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