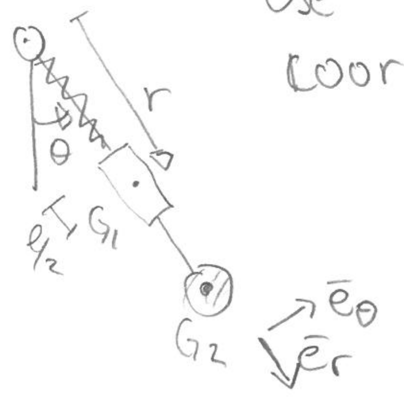
1



Use r, θ as generalized coordinates.

Kinetic energy for the pipe is $T_1 = \frac{m}{2}|\bar{v}_{G_1}|^2 + J_1\frac{\dot{\theta}^2}{2}$ with $\bar{v}_{G_1} = \dot{r}\bar{e}_r + r\dot{\theta}\bar{e}_\theta$ and $J_1 = \frac{m\ell^2}{12}$

Kinetic energy for the pipe is $T_2 = \frac{m}{2}|\bar{v}_{G_2}|^2$ with $\bar{v}_{G_2} = 3\ell\dot{\theta}\bar{e}_\theta$.

$$T = T_1 + T_2 = \frac{m}{2}\left(\dot{r}^2 + \left(r^2 + \frac{109}{12}\ell^2\right)\dot{\theta}^2\right)$$

Potential energy: $V_1^g = -mgr\cos\theta \qquad V_2^g = -mg3\ell\cos\theta$

$$V^k = \frac{k}{2}\Big(\underbrace{r - \frac{\ell}{2}}_{\text{pipe end}} - \underbrace{\ell}_{\text{natural length}}\Big)^2 = \frac{mg}{\ell}\left(r - \frac{3\ell}{2}\right)$$

$$V = V_1^g + V_2^g + V^k = \frac{mg}{\ell}\left[-\ell(r+3\ell)\cos\theta + \left(r - \frac{3\ell}{2}\right)^2\right]$$

Equlibrium points r_0, θ_0 satisfies

$$\left.\frac{\partial V}{\partial r}\right|_{r_0,\theta_0} = \frac{mg}{\ell}\left[-\ell\cos\theta_0 + 2\left(r_0 - \frac{3\ell}{2}\right)\right] = 0$$

$$\left.\frac{\partial V}{\partial \theta}\right|_{r_0,\theta_0} = \frac{mg}{\ell}\,\ell(r_0+3\ell)\sin\theta_0 = 0$$. We find $r_0 = 2\ell,\ \theta_0 = 0$ or $(r_0 = \ell,\ \theta_0 = \pi)$. Only the first is likely to be stable.

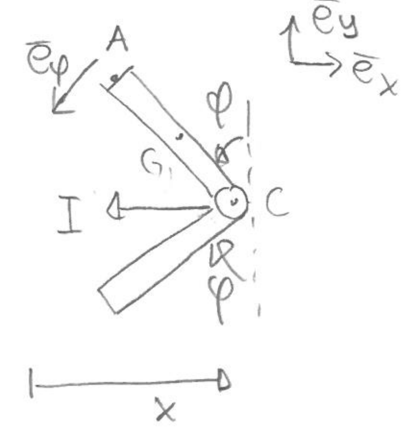
The mass matrix is $\mathbb{M} = \left.\frac{\partial^2 T}{\partial(\dot{r},\dot{\theta})^2}\right|_{r_0,\theta_0} = m\begin{bmatrix} 1 & 0 \\ 0 & \frac{157}{12}\ell^2 \end{bmatrix}$

The stiffness matix is $\mathbb{K} = \left.\frac{\partial^2 V}{\partial(r,\theta)^2}\right|_{r_0,\theta_0} = \frac{mg}{\ell}\begin{bmatrix} 2 & 0 \\ 0 & 5\ell^2 \end{bmatrix}$

Since the matrices are diagonal we find

$\boxed{\omega_1^2 = \frac{2g}{\ell}}$ (radial osc) and $\boxed{\omega_2^2 = \frac{60}{157}\frac{g}{\ell}}$ (angular osc)

2]



By symmetry, the point C moves only in th x direction and both rods make the same angle φ.

Use x, φ as generalized coordinates. For the rod AC:

$$T_1 = \frac{m}{2}|\bar{v}_{G_1}|^2 + J_1 \frac{\dot{\varphi}^2}{2} \quad \text{with}$$

$$\bar{v}_{G_1} = \dot{x}\,\bar{e}_x + \frac{\ell}{2}\dot{\varphi}\,\bar{e}_\varphi \quad \text{and } J_1 = \frac{m\ell^2}{12}$$

Using $\bar{e}_x \cdot \bar{e}_\varphi = -\cos\varphi$ we find $T_1 = \frac{m}{2}\left(\dot{x}^2 - \ell\cos\varphi\,\dot{x}\dot{\varphi} + \frac{\ell^2}{4}\dot{\varphi}^2\right) + \frac{m\ell^2}{12}\frac{\dot{\varphi}^2}{2}$

$$T = T_1 + T_2 = 2T_1 = m\left[\dot{x}^2 - \ell\cos\varphi\,\dot{x}\dot{\varphi} + \frac{\ell^2}{3}\dot{\varphi}^2\right]$$

Since $\bar{v}_C = \bar{e}_x\dot{x} + \bar{0}\dot{\varphi}$ we find the generalized impulses $I_x = \bar{e}_x \cdot (-I\bar{e}_x) = -I$, $I_\varphi = \bar{0}\cdot(-I\bar{e}_x) = 0$

Lagrange's equations for impact are

$$\left.\frac{\partial T}{\partial \dot{x}}\right|_{\dot{x}_f,\dot{\varphi}_f} - \left.\frac{\partial T}{\partial \dot{x}}\right|_{\dot{x}_i,\dot{\varphi}_i} = I_x \qquad \left.\frac{\partial T}{\partial \dot{\varphi}}\right|_{\dot{x}_f,\dot{\varphi}_f} - \left.\frac{\partial T}{\partial \dot{\varphi}}\right|_{\dot{x}_i,\dot{\varphi}_i} \qquad \text{with } \dot{x}_i = v_0,\ \dot{\varphi}_i = 0$$

The second equation is

$$m\left[-\ell\cos\varphi_0\,\dot{x}_f + \frac{2\ell^2}{3}\dot{\varphi}_f\right] - m\left[-\ell\cos\varphi_0 v_0 + \frac{2\ell^2}{3}\cdot 0\right] = 0$$

$\Rightarrow \dot{\varphi}_f = \frac{3}{2}\frac{\cos\varphi_0}{\ell}(\dot{x}_f - v_0)$ (*). Elastic impact means $\dot{x}_f = -v_0$.

$\Rightarrow \dot{\varphi}_f = -\frac{3\cos\varphi_0 v_0}{\ell}$. This gives $\bar{v}_{A_f} = \dot{x}_f\bar{e}_x + \ell\dot{\varphi}_f\bar{e}_\varphi =$

$$= -v_0\bar{e}_x - 3\cos\varphi_0 v_0\bar{e}_\varphi = v_0(-\bar{e}_x - 3\cos\varphi_0\bar{e}_\varphi) =$$

$$= \boxed{v_0\left[(-1 + 3\cos^2\varphi_0)\bar{e}_x + 3\sin\varphi_0\cos\varphi_0\bar{e}_y\right]}.$$

Instead of using $\dot{x}_f = -v_0$ we could have used $T_f - T_i = \{(*)\} = m\left[1 - \frac{3}{4}\cos^2\varphi_0\right](\dot{x}_f^2 - v_0^2) = 0$ to conclude that $\dot{x}_f = -v_0$.

3

Use cylindrical coordinates:

$$\bar{v}_G = (R-a)\dot{\theta}\bar{e}_\theta + \dot{z}\bar{e}_z$$

$$\bar{\omega} = \omega_r \bar{e}_r + \omega_\theta \bar{e}_\theta + \omega_z \bar{e}_z$$

$$\bar{p} = m\bar{v}_G = m\left[(R-a)\dot{\theta}\bar{e}_\theta + \dot{z}\bar{e}_z\right]$$

$$\bar{L}_G = \{\text{sperical symmetry}\} = J_G\bar{\omega} = \frac{2ma^2}{5}\bar{\omega}$$

Call the contact force $\bar{R} = -N\bar{e}_r + \bar{F}_\mu$.

Momentum balance:

$$\dot{\bar{p}} = -mg\bar{e}_z + \bar{R} \quad (1)$$

Angular momentum balance:

$$\dot{\bar{L}}_G = \bar{r}_{GC} \times \bar{R} = a\bar{e}_r \times \bar{R} \quad (2)$$

Eliminate $\bar{R}$:

$$(2) - \bar{r}_{GC} \times (1) \Rightarrow \dot{\bar{L}}_G - a\bar{e}_r \times \dot{\bar{p}} = a\bar{e}_r \times mg\bar{e}_z = -mga\bar{e}_\theta$$

Either use the known $\dot{\bar{e}}_r = \dot{\theta}\bar{e}_\theta$, $\dot{\bar{e}}_\theta = -\dot{\theta}\bar{e}_r$ or

use $\frac{d\bar{A}}{dt} = \left(\frac{d\bar{A}}{dt}\right)_{r,\varphi,z} + \bar{\omega}_{r,\varphi,z} \times \bar{A}$ with $\bar{\omega}_{r,\varphi,z} = \dot{\theta}\bar{e}_z$

to get

$$\dot{\bar{L}}_G - a\bar{e}_r \times \dot{\bar{p}} = \frac{2ma^2}{5}\left(\dot{\omega}_r\bar{e}_r + \dot{\omega}_\theta\bar{e}_\theta + \dot{\omega}_z\bar{e}_\theta + \omega_r\dot{\theta}\bar{e}_\theta - \omega_\theta\dot{\theta}\bar{e}_r\right) -$$

$$- a\bar{e}_r \times m\left((R-a)\ddot{\theta}\bar{e}_\theta + \ddot{z}\bar{e}_z - (R-a)\dot{\theta}^2\bar{e}_r\right) = -mga\bar{e}_\theta$$

We find

$$\bar{e}_r: \quad \frac{2ma^2}{5}(\dot{\omega}_r - \dot{\theta}\omega_\theta) = 0 \quad (3)$$

$$\bar{e}_\theta: \quad \frac{2ma^2}{5}(\dot{\omega}_\theta + \dot{\theta}\omega_r) + ma\ddot{z} = -mga \quad (4)$$

$$\bar{e}_z: \quad \frac{2ma^2}{5}\dot{\omega}_z - ma(R-a)\ddot{\theta} = 0 \quad (5)$$

3 cont.

Then we have the rolling condition:

$$\bar{v}_G + \bar{\omega} \times \bar{r}_{GC} = \bar{0} \Rightarrow (R-a)\dot{\theta}\bar{e}_\theta + \dot{z}\bar{e}_z + (\omega_r \bar{e}_r + \omega_\theta \bar{e}_\theta + \omega_z \bar{e}_z) \times a\bar{e}_r = \bar{0}$$

$\bar{e}_r$: $0 = 0$

$\bar{e}_\theta$: $(R-a)\dot{\theta} + a\omega_z = 0$ (6)

$\bar{e}_z$: $\dot{z} - a\omega_\theta = 0$ (7)

(3)-(7) determine the motion.

$\frac{d}{dt}$(6) in (5) gives $\ddot{\theta} = 0$ so $\boxed{\dot{\theta}(t) = \dot{\theta}(0) = \Omega}$

$\boxed{\omega_z(t) = -\frac{R-a}{a}\Omega}$ for some constant Ω.

Assuming constant z, (7) $\Rightarrow \omega_\theta = 0$

Using this in (3), (4) gives

$\dot{\omega}_r = 0$, $\frac{2ma^2}{5}\Omega\omega_r = -mga$ so if $\Omega \neq 0$ we

get a constant z solution

$$\boxed{\omega_r(t) = -\frac{5g}{2a\Omega}, \quad \omega_\theta(t) = 0, \quad z(t) = z_0}$$

3 cont) General solution:

If $\Omega = 0$ we find $\ddot{\theta}=0$, $\dot{\omega}_r = 0$, $\ddot{z} = -\frac{5}{7}g$ but this means rolling without normal force, which seems rather unphysical, and any disturbance will turn it into free fall $\ddot{z} = -g$.

If $\Omega \neq 0$ we use $\frac{d}{dt}(7)$ to eliminate $\ddot{z}$ from (4) and $\frac{d}{dt}(3)$ to eliminate $\dot{\omega}_\theta$ from (4) to get

$$\ddot{\omega}_r + \frac{2}{7}\Omega^2 \omega_r = -\frac{5}{7}\frac{g\Omega}{a} \Rightarrow \left\{ \omega_n = \sqrt{\frac{2}{7}}\,\Omega \right\}$$

$$\omega_r(t) = A\cos(\omega_n t) + B\sin(\omega_n t) - \frac{5g}{2a\Omega}$$

$$z(t) = \left\{ (7), (3) \Rightarrow \dot{z} = \frac{a\dot{\omega}_r}{\Omega} \right\} = \frac{a}{\Omega}\omega_r(t) + C_1$$

Thus z will oscillate vertically, while θ changes linearly in time.

4] The point A is also the center of mass. The x, y, z axes are principal axes for $\bar{\bar{J}}_A$ and they are equivalent, so $\bar{\bar{J}}_A$ is spherical and

$$\mathbb{J}_A = J_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with } J_A = \mathbb{J}_{A_{xx}} = 0 + \frac{ml^2}{12} + \frac{ml^2}{12} = \frac{ml^2}{6}$$

Moving to the point O, we get

$$\mathbb{J}_0 = \mathbb{J}_A + 3m \begin{bmatrix} y_A^2 + z_A^2 & -x_A y_A & -x_A z_A \\ -y_A x_A & x_A^2 + z_A^2 & -y_A z_A \\ -z_A x_A & -z_A y_A & y_A^2 + z_A^2 \end{bmatrix} =$$

$$= \frac{ml^2}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{3ml^2}{4} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} =$$

$$\boxed{= ml^2 \begin{bmatrix} 5/3 & -3/4 & -3/4 \\ -3/4 & 5/3 & -3/4 \\ -3/4 & -3/4 & 5/3 \end{bmatrix}}$$

Since $\bar{\bar{J}}_A$ is spherical, the $\boxed{(\bar{e}_x + \bar{e}_y + \bar{e}_z)}$ axis is (inertiawise) an axis of rotation symmerty, and we find $\mathbb{J}_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{ml^2}{6} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ so $\boxed{J_1 = \frac{ml^2}{6}}$

We also know from rotation symmetry that $J_2 = J_3$ for any two axes perpendicular to $\bar{e}_x + \bar{e}_y + \bar{e}_z$, for example $\bar{e}_x - \bar{e}_z$ and $\bar{e}_x - 2\bar{e}_y + \bar{e}_z$.

Also $J_1 + J_2 + J_3 = \text{Trace}(\mathbb{J}_0) = 5ml^2$

$$\boxed{J_2 = J_3 = \frac{29}{12} ml^2}$$

5] $T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) \quad V = \frac{k}{2}(r-\ell)^2$

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{2}(r-\ell)^2$$

$$\frac{\partial L}{\partial \theta} = 0 \Rightarrow \theta \text{ is cyclic} \Rightarrow p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \quad (1)$$

The r Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = m(\ddot{r} - r\dot{\theta}^2) + k(r-\ell) = 0 \quad (2)$$

Assuming a relative eq point $r(t) = r_0 \quad \dot{\theta}(t) = \Omega_0$

we find $p_\theta = m r_0^2 \Omega_0$ and $-m r_0 \Omega_0^2 + k(r_0 - \ell) = 0$

$\Rightarrow \boxed{\Omega_0^2 = \frac{k}{m}\frac{r_0 - \ell}{r_0}, \quad r_0 \geq \ell}$. This also gives

$$\boxed{(p_\theta)_0^2 = m^2 r_0^4 \Omega_0^2 = m^2 r_0^4 \frac{k}{m}\frac{r_0-\ell}{r_0} = k m r_0^3 (r_0 - \ell).}$$

We can now use (1) to eliminate $\dot{\theta}^2$ from (2):

$(1) \Rightarrow \dot{\theta}^2 = \frac{(p_\theta)_0^2}{m^2 r^4} = \frac{k}{m}\frac{r_0^3}{r^4}(r_0 - \ell)$. Insert into (2):

$$m\left(\ddot{r} - \frac{k}{m}\frac{r_0^3}{r^3}(r_0 - \ell)\right) + k(r-\ell) = 0 \Rightarrow$$

$$\ddot{r} + \frac{k}{m}\underbrace{\left[(r-\ell) - \frac{r_0^3}{r^3}(r_0-\ell)\right]}_{f(r)} = 0.$$ Clearly $f(r_0) = 0$ and

$f'(r_0) = 1 + \frac{3(r_0-\ell)}{r_0}$. The linarized eqation is

$$\ddot{r} + \underbrace{\frac{k}{m}\left(1 + \frac{3(r_0-\ell)}{r_0}\right)}_{\omega^2}(r - r_0) = \mathcal{O}(r-r_0)^2$$ and

we find the oscillation frequency $\boxed{\omega^2 = \frac{k}{m}\left(1 + 3\frac{(r_0-\ell)}{r_0}\right)}$

or in terms of the rotation speed Ω_0

$\omega^2 = \frac{k}{m} + 3\Omega_0^2$. [Compare with problem 23 in the problem collection, but where rotation speed is fixed.]

6)

$$\delta \int_{t_0}^{t_1} \left[\left(\sum_a \dot{q}_a p_a \right) - H(q, p, t) \right] dt = \{ t_0, t_1 \text{ fixed} \} =$$

$$= \int_{t_0}^{t_1} \sum_a \left(\delta \dot{q}_a p_a + \dot{q}_a \delta p_a - \frac{\partial H}{\partial q_a} \delta q_a - \frac{\partial H}{\partial p_a} \delta p_a \right) dt =$$

$$= \{ \text{partial integration on } \delta \dot{q}_a p_a \} =$$

$$= \Big[\sum_a (\delta q_a p_a) \Big]_{t_0}^{t_1} + \int_{t_0}^{t_1} \sum_a \left(-\delta q_a \dot{p}_a + \dot{q}_a \delta p_a - \frac{\partial H}{\partial q_a} \delta q_a - \frac{\partial H}{\partial p_a} \delta p_a \right) dt$$

$$= \{ q_a(t_1), q_a(t_0) \text{ fixed} \} =$$

$$= \int_{t_0}^{t_1} \sum_a \left[\delta q_a \left(-\dot{p}_a - \frac{\partial H}{\partial q_a} \right) + \delta p_a \left(\dot{q}_a - \frac{\partial H}{\partial p_a} \right) \right] dt$$

By the varying each component of δq_a, δp_a independently, and using the standard argument (see course text) we find $-\dot{p}_a - \frac{\partial H}{\partial q_a} = 0$ $\dot{q}_a - \frac{\partial H}{\partial p_a} = 0$ for $t_0 \leq t \leq t_1$.