Lecture 3
Channel Equalization

M. Xiao
CommTh/EES/KTH

Lecture 3: Channel Equalization 3
Advanced Digital Communications (EQ2410)[1]

M. Xiao
CommTh/EES/KTH

Monday, Jan. 25, 2016
15:00-17:00, B23

[1] Textbook: U. Madhow, *Fundamentals of Digital Communications*, 2008

Notes

Lecture 3
Channel Equalization

M. Xiao
CommTh/EES/KTH

Overview

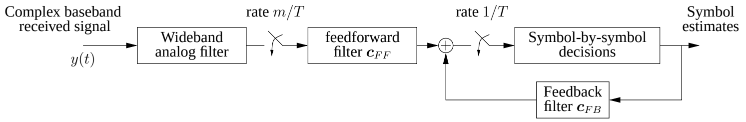
Lecture 1+2

- Channel models
- Optimal receiver design and ML sequence estimation
- Linear equalization

Lecture 3: Decision Feedback Equalization and Performance Evaluation

Notes

Decision Feedback Equalization



- Drawback of linear EQ: noise enhancement (especially for ZF but as well for MMSE)
- Decision feedback equalization (DFE)
  - Uses feedback from prior decisions to cancel interference due to past symbols;
  - Linearly suppresses interference from future symbols.
- Design of the feedforward correlator $\mathbf{c}_{FF}$
  - Design methods for linear equalization (ZF or MMSE) are applied to the reduced model

$$\mathbf{r}_n^f = b[n]\mathbf{u}_0 + \sum_{j>0} b[n+j]\mathbf{u}_j + \mathbf{w}[n] = \mathbf{U_f}\mathbf{b}_f[n] + \mathbf{w}[n].$$

→ Only the ISI from future symbols is considered.
→ Replace $\mathbf{U}$ in the design method of the linear equalizer by $\mathbf{U}_f = [\mathbf{u}_0, \dots, \mathbf{u}_{k_2}]$.

Notes

Decision Feedback Equalization

- Output of the feedforward correlator

$$\mathbf{c}_{FF}^H\mathbf{r}[n] = b[n]\mathbf{c}_{FF}^H\mathbf{u}_0 + \left\{\sum_{j>0} b[n+j]\mathbf{c}_{FF}^H\mathbf{u}_j\right\} + \mathbf{c}_{FF}^H\mathbf{w}[n] + \sum_{j>0} b[n-j]\mathbf{c}_{FF}^H\mathbf{u}_{-j}$$

→ The feedforward correlator suppresses the ISI from future symbols (i.e., the term within $\{\}$).
→ Decision feedback (i.e., previous symbol estimates $\hat{b}[n-1], \hat{b}[n-2], \dots$) is used to cancel the last term.

- With $c_{FB}[j] = -\mathbf{c}_{FF}^H\mathbf{u}_{-j}$, we get the DFE decision variable

$$\begin{aligned} Z_{DFE}[n] &= \mathbf{c}_{FF}^H\mathbf{r}[n] + \sum_{j>0} c_{FB}[j]\hat{b}[n-j] \\ &= b[n]\mathbf{c}_{FF}^H\mathbf{u}_0 + \left\{\sum_{j>0} b[n+j]\mathbf{c}_{FF}^H\mathbf{u}_j\right\} + \mathbf{c}_{FF}^H\mathbf{w}[n] + \\ &\quad \sum_{j>0}(b[n-j] - \hat{b}[n-j])\mathbf{c}_{FF}^H\mathbf{u}_{-j} \end{aligned}$$

- Matrix formulation of the feedback correlator: $\mathbf{c}_{FB} = -\mathbf{c}_{FF}^H\mathbf{U}_p$, with $\mathbf{U}_p = [\mathbf{u}_{k_1}, \dots, \mathbf{u}_{-1}]$ and $\mathbf{U} = [\mathbf{U}_p \ \ \mathbf{U}_f]$.

Notes

Lecture 3
Channel Equalization

M. Xiao
CommTh/EES/KTH

Performance of the DFE

- Problem: error propagation in the decision feedback.
- If the feedback is correct, performance is similar to the linear feedforward equalizer on the reduced model (error probability $P_{e,FF}$).
- Characterization of error propagation:
  - Error propagation event starts with the first decision error and ends if the detector is error free (after L_{FB} consecutive correct decisions)
  - Duration of one error event: T_e; number of symbol errors: N_e
  - Time between error events: T_c $\rightarrow$ geometric random variable

$$P[T_c = k] = P_{e,FF}(1 - P_{e,FF})^{k-1}$$

- Approximation for the error probability (with $\mathrm{E}[T_c] = 1/P_{e,FF}$)

$$P_{e,DFE} = \frac{\mathrm{E}[N_e]}{\mathrm{E}[T_e] + \mathrm{E}[T_c]} \approx \mathrm{E}[N_e]P_{e,FF}$$

(Approximation: $\mathrm{E}[T_e] \ll \mathrm{E}[T_c]$.)

$\rightarrow$ Since $\mathrm{E}[N_e]$ is typically small, the performance of the DFE is mainly characterized by the performance of the linear feedforward equalizer based on the reduced model.

Notes

Lecture 3
Channel Equalization

M. Xiao
CommTh/EES/KTH

Performance of the MLSE
– Assumptions and Definitions

- Real-valued signals and BPSK modulation with $b[n] \in \{-1, +1\}$.
- Continuous-time system model

$$y(t) = \sum_n b[n]p(t - nT) + n(t) = s(\mathbf{b}) + n(t)$$

with $s(\mathbf{b}) = s_{\mathbf{b}}(t) = \sum_n b[n]p(t - nT)$.

- Error probability for the k-th bit under ML detection

$$P_e(k) = \Pr(\hat{b}_{ML}[k] \neq b[k])$$

- Definition 5.8.1, *error sequence* corresponding to a sequence $\mathbf{b}$ and its estimate $\hat{\mathbf{b}}$: $\mathbf{e} = (\mathbf{b} - \hat{\mathbf{b}})/2$ (with $e[n] \in \{-1, 0, +1\}$).
- Consistency condition: if $e[n] \neq 0$ (i.e., $\hat{b}[n] \neq b[n]$), then $e[n] = b[n]$.
- Definition 5.8.2, an error sequence $\mathbf{e}$ is a *valid error sequence* for $\mathbf{b}$ if the consistency condition is satisfied for all elements in $\mathbf{e}$.

Notes

Performance of the MLSE
– Union Bound

We derive the union bound on the error probability in four steps:

Step 1: Decomposition of $P_e(k)$

$$\begin{aligned}
P_e(k) &= \Pr(b[k] \neq \hat{b}_{ML}[k]) \\
&= \sum_{\mathbf{e}\in\mathcal{E}} \sum_{\mathbf{b}\in\mathcal{B}} \Pr(b[k] \neq \hat{b}_{ML}[k], \mathbf{e}, \mathbf{b}) \\
&= \sum_{\mathbf{e}\in\mathcal{E}_k} \sum_{\mathbf{b}\in\mathcal{B}} \Pr(\hat{\mathbf{b}}_{\mathbf{ML}} = \mathbf{b} - 2\mathbf{e}, \mathbf{e} \text{ is valid for } \mathbf{b}, \mathbf{e}, \mathbf{b}) \\
&= \sum_{\mathbf{e}\in\mathcal{E}_k} \sum_{\mathbf{b}\in\mathcal{B}} \Pr(\hat{\mathbf{b}}_{\mathbf{ML}} = \mathbf{b} - 2\mathbf{e} | \mathbf{e} \text{ is valid for } \mathbf{b}, \mathbf{e}, \mathbf{b}) \cdot \\
&\qquad \cdot \Pr(\mathbf{e} \text{ is valid for } \mathbf{b}, \mathbf{e}, \mathbf{b}) \\
&\leq \sum_{\mathbf{e}\in\mathcal{E}_k} Q\left(\frac{\|s(\mathbf{e})\|}{\sigma}\right) \sum_{\mathbf{b}\in\mathcal{B}} \Pr(\mathbf{e} \text{ is valid for } \mathbf{b}, \mathbf{e}, \mathbf{b})
\end{aligned}$$

where

- $\mathcal{E}$ denotes the set of all existing error sequences $\mathbf{e}$.
- $\mathcal{E}_k$ denotes the set of all existing error sequences $\mathbf{e}$ which lead to an error event at symbol position k.
- $\mathcal{B}$ denotes the set of all possible symbol sequences $\mathbf{b}$.

Notes

Performance of the MLSE
– Union Bound

Step 2: Probability for a valid error sequence

$$\begin{aligned}
\sum_{\mathbf{b}\in\mathcal{B}} &\Pr(\mathbf{e} \text{ is valid for } \mathbf{b}, \mathbf{e}, \mathbf{b}) \\
&= \sum_{\mathbf{b}\in\mathcal{B}} \Pr(\mathbf{e} \text{ is valid for } \mathbf{b}, \mathbf{e}|\mathbf{b}) \Pr(\mathbf{b}) \\
&= \sum_{\mathbf{b}\in\mathcal{B}} \prod_{i=1}^{N} \Pr(e[i], e[i] \text{ is valid for } b[i] \,|\, b[i]) \Pr(b[i]) \\
&= \sum_{\substack{\mathbf{b}\in\mathcal{B},\\ \mathbf{e} \text{ is valid for } \mathbf{b}}} \prod_{\substack{i=1,\\ e[i]=0}}^{N} \Pr(b[i]) \prod_{\substack{i=1,\\ e[i]\neq 0}}^{N} \Pr(b[i] = e[i]) \\
&= \prod_{\substack{i=1,\\ e[i]\neq 0}}^{N} \Pr(b[i] = e[i]) \sum_{\substack{\mathbf{b}\in\mathcal{B},\\ \mathbf{e} \text{ is valid for } \mathbf{b}}} \prod_{\substack{i=1,\\ e[i]=0}}^{N} \Pr(b[i]) \\
&= 2^{-w(\mathbf{e})} \sum_{\substack{\mathbf{b}\in\mathcal{B},\\ \mathbf{e} \text{ is valid for } \mathbf{b}}} 2^{-(N-w(\mathbf{e}))} \\
&= 2^{-w(\mathbf{e})}
\end{aligned}$$

Notes

Performance of the MLSE
– Union Bound

Step 3: Pairwise error probability conditioned on an error sequence

- Pairwise error probability for two signals $s_1(t)$ and $s_2(t)$ in Gaussian noise:

$$P_{e,pw} = Q\left(\frac{\|s_1 - s_2\|}{2\sigma}\right).$$

- Pairwise error probability for detecting the signal $s(\hat{\mathbf{b}}_{\mathbf{ML}})$ assuming that $s(\mathbf{b})$ was transmitted:

$$\Pr(\hat{\mathbf{b}}_{\mathbf{ML}} = \mathbf{b} - 2\mathbf{e} | \mathbf{e} \text{ is valid for } \mathbf{b}, \mathbf{e}, \mathbf{b}) \leq Q\left(\frac{\|s(\mathbf{e})\|}{\sigma}\right).$$

- Note that $s(\mathbf{b}) - s(\mathbf{b} - 2\mathbf{e}) = 2s(\mathbf{e})$ due to linearity of the modulation format.

Step 4: Combining the results

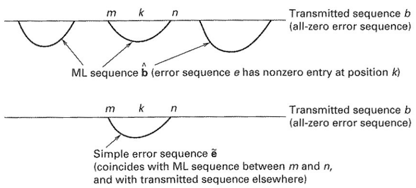
$$P_e(k) \leq \sum_{\mathbf{e} \in \mathcal{E}_k} Q\left(\frac{\|s(\mathbf{e})\|}{\sigma}\right) 2^{-w(\mathbf{e})}.$$

Notes

Lecture 3
Channel Equalization
M. Xiao
CommTh/EES/KTH

Performance of the MLSE
– Intelligent Union Bound

- Goal: consider only the most relevant terms in the summation in the union bound.
- Tool:
  Error sequence trellis



- Sequences $\hat{\mathbf{b}}$ are represented by their error sequences $\mathbf{e}$ relative to the transmitted sequence $\mathbf{b}$ (i.e., $\mathbf{b}$ corresponds to the all-zero path).
- Trellis state for channel with memory L

$$s_e[n] = [e[n-L], \ldots, e[n-1]] \quad \rightarrow \quad 3^L \text{ trellis states}$$

- Error sequences
  - If an error occurs, the path diverges from the all-zero path.
  - An error sequence merges again with the all-zero path if there are L consecutive zeros in the error sequence.

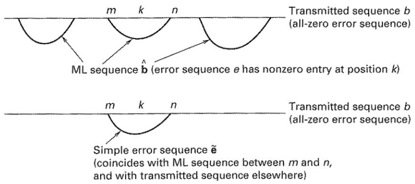
Notes

Lecture 3
Channel Equalization

M. Xiao
CommTh/EES/KTH

Performance of the MLSE
– Intelligent Union Bound

- Simple error sequence: an error sequence $\mathbf{e}$ is simple if there are no more than $L-1$ zeros between any two nonzero elements in $\mathbf{e}$.



- Set of simple error sequences with $e[k] \neq 0$: S_k
- Intelligent union bound using simple error sequences

$$P_e(k) \leq \sum_{\mathbf{e} \in S_k} Q\left\{\frac{\|s(\mathbf{e})\|}{\sigma}\right\} 2^{-w(\mathbf{e})}$$

- High-SNR approximation: consider only the contribution from the Q-function with the smallest argument $\epsilon_{\min}^2 = \min_{\mathbf{e}} \|s(\mathbf{e})\|^2$:

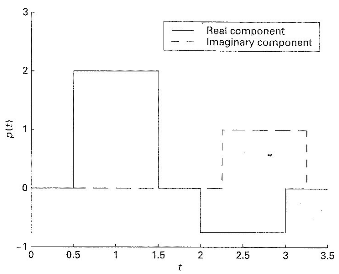
$$P_e(k) \sim exp\left(-\frac{\epsilon_{\min}^2}{2\sigma^2}\right)$$

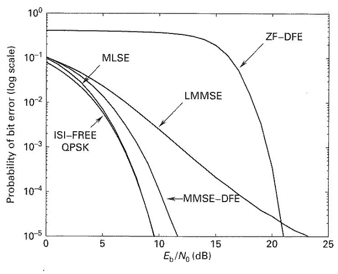
Notes

Lecture 3
Channel Equalization

M. Xiao
CommTh/EES/KTH

Comparison





Simulation Parameters

- QPSK, 1 symbol per time unit, rectangular transmit pulse $g_T(t)$
- Channel memory $L = 2$ (two echos)
- Receiver uses the optimal matched filter
  - → MLSE: $4^2 = 16$ states
  - → L-MMSE: observation interval of length $2L + 1 = 5$[2]
  - → DFE: 4 feedback taps

[2]Note that the L-ZF equalizer does not exist for this choice.

Notes

Lecture 3
Channel Equalization

M. Xiao
CommTh/EES/KTH

Summary

MLSE Equalization

- Optimal equalizer for finding the maximum likelihood sequence.
- Efficient implementation with Viterbi Algorithm.
  (Trellis states are give by the content of the memory of the channel.)
- Complex for large constellations and channels with large memory.
- Used for example in the GSM standard.

Linear Equalization

- Suboptimal equalizer with low complexity.
- Each symbol is detected by correlating an observation vector with a correlator
  - Zero-forcing (ZF): correlator is designed to completely erase ISI (if possible);
  - MMSE: minimize the mean squared error (MMSE) between the decision variable and the symbol.
- Drawback: noise enhancement (especially for the ZF equalization)
- Often used if MLSE is not applicable (large constellations, channels with large memory), for example underwater communications or mobile radio communications with low-complexity receivers.

Notes

Lecture 3
Channel Equalization

M. Xiao
CommTh/EES/KTH

Summary

Decision Feedback Equalization (DFE)

- Non-linear suboptimal equalizer with low/moderate complexity.
- Decision feedback of previous symbol estimates are used to cancel ISI from past symbols;
- Under the assumption that the decision feedback is correct, a linear equalizer is used to suppress the ISI from future symbols
- Drawback: error propagation due to wrong decisions.
- Often used in underwater communications; similar techniques can be used for reduced-state MLSE.

Notes