Plane waves, Fourier transforms, Generalised functions and Greens functions

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Outline

- Plane waves
  - Phase velocity and eigenmodes
  - Relation to Fourier series and Fourier transforms
- Fourier transforms of generalised functions
  - Plemej formula
- Laplace transforms and complex frequencies
  - Theorem of residues
  - Causal functions
  - Relations between Laplace and Fourier transforms
- Greens functions
  - Poisson equation
  - d' Alemberts equation
  - Wave equations in temporal gauge

Plane waves

- Plane waves have the form

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$$

  - With wave number k and frequency ω.
  - Why the name plane waves?

- For $|\mathbf{k}| = \omega$, plane waves are solutions to d'Alembert's equation

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E}$$

- Plane waves are also solution's to Maxwell's equation in vacuum

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla\times(\nabla\times\mathbf{E})$$

  if and only if $\mathbf{k}\cdot\mathbf{E} = 0$.
  - i.e. Maxwell's equations in vacuum only allow *transverse waves*!

Phase velocity

- The velocity of a waves front is called the *phase velocity*.

- At $t = 0$ and $\mathbf{x} = \mathbf{0}$ the phase of the plane wave is: $i\mathbf{k} \cdot \mathbf{x} - i\omega t = 0$
- Where is the corresponding wave front at $t = dt$?

- Assume, $\mathbf{k} = k_x \mathbf{e}_x$, then the wave propagates along the x-axis
- Denote x at the new wave front by dx, then

$$ik_x dx - i\omega dt = 0$$

$$dx = \frac{\omega}{k_x} dt$$

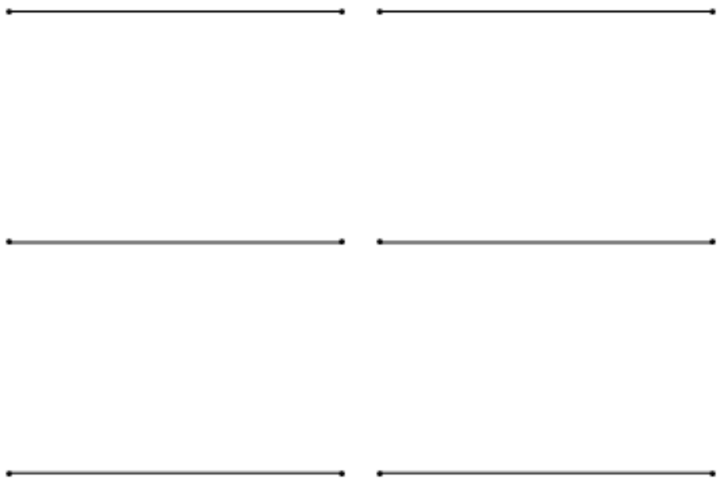
- Thus the phase velocity of a plane wave is thus

$$\left|\mathbf{v}_{ph}\right| = \frac{|\omega|}{|\mathbf{k}|} \quad \text{or} \quad \mathbf{v}_{ph} = \frac{\omega}{|\mathbf{k}|} \frac{\mathbf{k}}{|\mathbf{k}|}$$

What is the phase velocity of EM waves in vacuum?

Eigenmodes

- When you put a wave in a “box” it has to satisfy certain boundary conditions.
- *Example*: guitar strings



- The motion is constraint to oscillate only at certain *eigenmodes*, each having an *eigenfrequency*, ω_j, and $k_j = \omega_j / v_{ph}$:

$$E(x,t) \sim \sum_j E_j cos(k_j x$$

- How are eigenmodes related to plane waves?

Plane waves and Fourier transforms

- The sum over plane waves that solves a wave equation in a "box" is in fact a *Fourier series*
  - And the amplitudes are the *Fourier coefficients*

- In the infinite domain there are no boundary conditions to restrict the possible frequencies
  - All real frequencies are possible!
  - Sum over all real frequencies means an inverse Fourier transform!

- Fourier transform calculates an "*amplitude density*" in ω-space

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t} dt$$

- Inverse Fourier transform is a *sum over all frequencies*

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t} d\omega$$

Basic theory of Fourier transformed…

- The Fourier integral theorem:
  - *f(t)* is sectionally continuous over $-\infty < t < \infty$
  - *f(t)* is defined as $$f(t) = \lim_{\delta \to 0} \frac{1}{2}\left[f(t+\delta) + f(t-\delta) \right]$$
  - *f(t)* is *amplitude integrable*, that is, $$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

  Then the following identity holds:
  $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z) e^{\pm iy(z-t)} dz dy$$

- Do these function have a Fourier transform?

$$f(t) = 1$$
$$f(t) = \cos(t)$$
$$f(t) = \exp(-t)$$
$$f(t) = \begin{cases} 0 & , \quad t < 0 \\ \exp(-t) & , \quad t \geq 0 \end{cases}$$
$$f(t) = \begin{cases} 0 & , \quad t \text{ is rational number} \\ \exp(-t) & , \quad t \text{ is rational number} \end{cases}$$

What functions have a Fourier transform?

- We are interested in Fourier transform to represent plane waves
  - But plane waves don't have a Fourier transform!!

- **Solution**: Use approximations of $\cos(t)$ that converge asymptotically to $\cos(t)$ – details comes later on…

  - *NOTE*: The asymptotic limits of functions like $\cos(t)$ will be used to define generalised functions, e.g. Dirac d-function.

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  - Relation to Fourier series and Fourier transforms

- Fourier transforms of generalised functions ← For transforming $\cos(t)$
  - Plemej formula

- Laplace transforms and complex frequencies ← For transforming $\exp(t)$
  - Theorem of residues
  - Causal functions
  - Relations between Laplace and Fourier transforms

- Greens functions
  - Poisson equation
  - d' Alemberts equation
  - Wave equations in temporal gauge

Dirac δ-function

- Dirac's generalised function can be defined as:

$$\delta(x) = \begin{cases} 0 \,, & x \neq 0 \\ \infty \,, & x = 0 \end{cases} \qquad \& \qquad \int_{-\infty}^{\infty} \delta(x) = 1$$

Alternative definitions, as limits of well behaving functions, will be identified later!

- Important example:

$$\int_{-\infty}^{\infty} \delta(f(t))dt = \sum_{i:f(t_i)=0} \frac{1}{f'(t_i)}$$

Proof: Whenever $|f(t)|>0$ the contribution is zero. For each $t = t_i$ where $f(t_i)=0$, perform the integral over a small region $t_i - \varepsilon < t < t_i + \varepsilon$ (where ε is small such $f(t) \approx (t - t_i) f'(t_i)$). Next, use variable substitution to perform the integration in $x = f(t)$, then $dt = dx / f'(t_i)$:

$$\int_{-\infty}^{\infty} \delta(f(t))dt = \sum_{i:f(t_i)=0} \int_{-\infty}^{\infty} \frac{1}{f'(t_i)} \delta(x)dx = \sum_{i:f(t_i)=0} \frac{1}{f'(t_i)}$$

Truncations and Generalised functions

- To approximate the Fourier transform of $f(t) = 1$, use *truncation*.

Truncation of a function *f(t)*:

$$f_T(t) = \begin{cases} f(t), & |t| < T \\ 0, & |t| > T \end{cases}$$, such that $f(t) = \lim_{T\to\infty} f_T(t)$

- Then for $f(t) = 1$

$$\mathbf{F}\{f_T(t)\} = \int_{-\infty}^{\infty} f_T(t) e^{-i\omega t} dt = \int_{-T}^{T} 1 e^{-i\omega t} dt = \frac{\sin(\omega t/2)}{\omega/2}$$

  – When $T\to\infty$ then this function is zero everywhere except at $\omega = 0$ and its integral is 2π, i.e.

$$\mathrm{F}\{1\} = \lim_{T\to\infty} \frac{\sin(\omega T/2)}{\omega/2} = 2\pi\delta(\omega)$$

  – Note: $\mathbf{F}\{1\}$ exists *only* as an asymptotic of an ordinary function, i.e. a *generalised function*.

More generalised function

- An alternative to truncation is *exponential decay*

$$f_\eta(t) = f(t)e^{-\eta|t|} \quad \text{, such that} \quad f(t) = \lim_{\eta\to 0} f_\eta(t)$$

- Three important examples:
  - *f(t)=1* :

$$\mathbf{F}\{f_\eta(t)\} = \frac{2\pi\eta}{\omega^2+\eta^2} \quad \Longrightarrow \quad \mathrm{F}\{1\} = \lim_{\eta\to 0}\frac{2\pi\eta}{\omega^2+\eta^2} = 2\pi\delta(\omega)$$

  - Sign function, $\mathrm{sgn}(t)$: $\mathbf{F}\{\mathrm{sgn}(t)\} = \lim_{\eta\to 0}\mathbf{F}\{e^{-\eta|t|}\mathrm{sgn}(t)\} = \lim_{\eta\to 0}\frac{2i\omega}{\omega^2+\eta^2} = 2i\wp\left[\frac{1}{\omega}\right]$

  - The generalised function is the *Cauchy principal value function:*

$$\wp\frac{1}{\omega} := \lim_{\eta\to 0}\frac{\omega}{\omega^2+\eta^2} = \begin{cases} 1/\omega\,, & \text{for} \quad \omega \neq 0 \\ 0\,, & \text{for} \quad \omega = 0 \end{cases}$$

  - Heaviside function *f(t)=H(t)* : $\mathrm{F}\{H(t)\} = \lim_{\eta\to 0}\frac{i}{\omega+i\eta}$

    This generalised function is often written as: $\frac{1}{\omega+i0} := \lim_{\eta\to 0}\frac{1}{\omega+i\eta}$

Plemelj formula

- Relation between *H(t)* and *sgn(t)*:

$$2H(t) = 1 + \operatorname{sgn}(t)$$

with the Fourier transform:

$$\frac{1}{\omega + i0} = \wp \frac{1}{\omega} - i\pi\delta(\omega)$$

This is known as the *Plemelj formula*

– Note: How we treat $w = 0$ matters! …but why?

– We will use the Plemelj formula when describing resonant wave damping (see later lectures)

Driven oscillator with dissipation

- Example of the Plemelj formula: a driven oscillator with eigenfrequency Ω :

$$\frac{\partial^2 f(t)}{\partial t^2} + \Omega^2 f(t) = E(t)$$

with dissipation coefficient ν :

$$\frac{\partial^2 f(t)}{\partial t^2} + 2\nu \frac{\partial f(t)}{\partial t} + \Omega^2 f(t) = E(t)$$

- Fourier transform: $\left(-\omega^2 - i2\nu\omega + \Omega^2\right) f(\omega) = E(\omega)$

- Solution:

$$f(\omega) = \frac{E(\omega)}{-\omega^2 - i2\nu\omega + \Omega^2} = \frac{E(\omega)}{2\hat{\Omega}} \left[\frac{1}{\omega - \hat{\Omega} + i\nu} - \frac{1}{\omega + \hat{\Omega} + i\nu} \right]$$

where $\hat{\Omega} = \sqrt{\Omega^2 - \nu^2}$

- Take limit when damping ν goes to zero:

$$f(\omega) = \frac{E(\omega)}{2\Omega} \left[\frac{1}{\omega - \Omega + i0} - \frac{1}{\omega + \Omega + i0} \right]$$

use Plemelj formula

Later we'll look at the inverse transform

$$f(\omega) = \frac{E(\omega)}{2\Omega} \left[\wp\left(\frac{1}{\omega - \Omega} \right) - \wp\left(\frac{1}{\omega + \Omega} \right) - i\pi\delta\left(\omega - \Omega\right) + i\pi\delta\left(\omega + \Omega\right) \right]$$

Physics interpretation of Plemej formula

- For oscillating systems:
  eigenfrequency Ω will appear as *resonant denominator*

$$f(\omega) \sim \frac{1}{\omega \pm \Omega} \quad \Leftrightarrow \quad f(t) \sim e^{\pm i\Omega t}$$

Including infinitely small dissipation and applying Plemelj formula

$$\frac{1}{\omega - \Omega + i0} = \wp \frac{1}{\omega - \Omega} - i\pi\delta\left(\omega - \Omega\right)$$

- Later lectures on the dielectric response of plasma:
  When the dissipation goes to zero for a kinetic plasma there is still a wave damping called Landau damping, a “collisionless” damping, which comes from the δ-function

$$"damping" \sim i\pi\delta\left(\omega - \Omega\right)$$

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For transforming $\cos(t)$ (← Fourier transforms of generalised functions)

For transforming $\exp(t)$ (← Laplace transforms and complex frequencies)

Laplace transforms and complex frequencies (Chapter 8)

- Fourier transform is restricted to handling real frequencies,
  - Not optimal for damped or growing modes
  - For complex frequencies (damped/growing modes) we need the Laplace transforms!
- To understand better the relation between Fourier and Laplace transforms we will first study the residual theorem and see it applied to the Fourier transform of causal functions.

The Theorem of Residues

- Expand $f(z)$ around singularity, $z=z_i$:

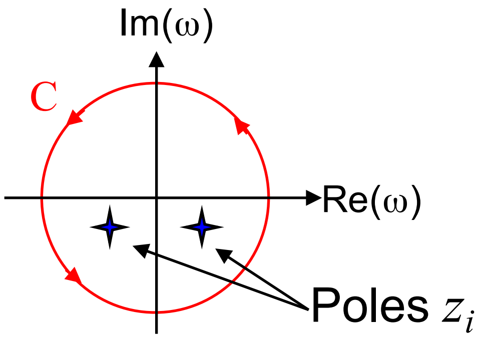
$$f(z) \approx \frac{R_i}{(z-z_i)} + c_0 + c_1(z-z_i) + \ldots$$

  - the point $z=z_i$ is called a *pole*
  - the numerator R_i is the *residue*

- The integral along closed contour in the complex plane can be solved using *the theorem of residues*

$$\int_C f(z)dz = 2\pi i \sum_i R_i$$

$$R_i = \lim_{z \to z_i} (z-z_i) f(z)$$



  - where the sum is over all poles z_i inside the contour

Example: Theorem of Residues

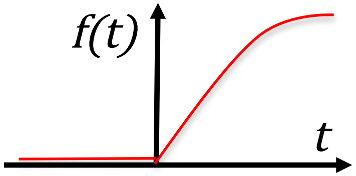
- **Example**: $f(z)=1/z$ and C encircling a poles at $z=0$

$$\int_C f(z)dz = \int_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{re^{i\theta}} ire^{i\theta} d\theta = \int_0^{2\pi} i d\theta = 2\pi i$$

$$where \quad z = re^{i\theta} \quad dz = ire^{i\theta} d\theta$$

Causal functions

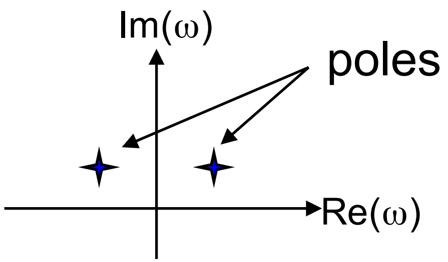
- ***Causal functions***:
  functions f_c that "start" at $t=0$, such that $f_c(t)=0$ for $t<0$.



- **Example**: causal damped oscillation $f_c(t) = e^{-\gamma t} \cos(\Omega t)$, for $t>0$

$$\mathbf{F}\{f_c(t)\} = \int_0^\infty e^{-iwt} e^{-\gamma t} \cos(\Omega t) dt = \frac{i}{2}\left[\frac{1}{\omega - \Omega - i\gamma/2} + \frac{1}{\omega + \Omega - i\gamma/2}\right]$$

  - The two denominators are poles in the complex ω plane
  - Both poles are in the upper half of the complex plane $\mathrm{Im}(\omega)<0$



- Causal function are suitable for Laplace transformations
  - to better understand the relation between Laplace and Fourier transforms; study the *inverse* Fourier transform of the causal damped oscillator

Causal functions and contour integration

- Use *residual theory* for the inverse Fourier transform

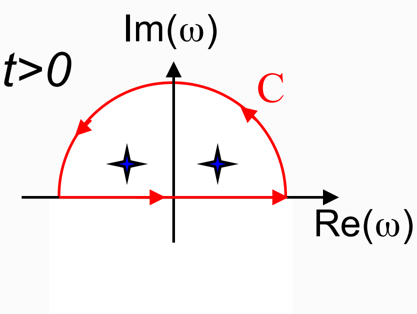
$$\mathbf{F}^{-1}\{f_c(t)\} = \frac{1}{2\pi}\int d\omega\, e^{i\omega t}\left\{\frac{i}{2}\left[\frac{1}{\omega-\Omega-i\gamma/2}+\frac{1}{\omega+\Omega-i\gamma/2}\right]\right\}$$

For $t > 0$:

- For $\mathrm{Im}(\omega) \to \infty$, then $e^{i\omega t} \to 0$ and $\lim_{|\omega|\to\infty} \tilde{f}_c(\omega) \sim 1/\omega \to 0$
- Thus, close contour with half circle $\mathrm{Im}(\omega)>0$
- Inverse Fourier transform is sum of *residues* from poles

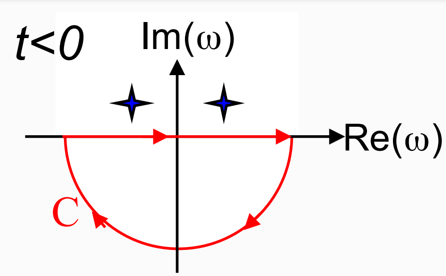
$$f_c(t) = \frac{1}{2\pi}\int_C e^{i\omega t}\frac{i}{2}\left[\frac{1}{\omega-\Omega+i\gamma/2}+\frac{1}{\omega+\Omega+i\gamma/2}\right]d\omega$$

$$= -\sum_i iR_i = -i\frac{i}{2}\left[e^{(i\Omega-\gamma/2)t}+e^{(-i\Omega-\gamma/2)t}\right]$$



For $t < 0$:

- $e^{i\omega t} \to 0$, for $\mathrm{Im}(\omega) \to -\infty$;
  close contour with half circle $\mathrm{Im}(\omega)<0$
- No poles inside contour: $f(t)=0$ for $t<0$



Laplace transform

- Laplace transform of function *f(t)* is

$$F(s) = L\{f(t)\} = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\infty} e^{-st} f(t) dt$$

  – Like a Fourier transform for a causal function, but $i\omega \to s$.

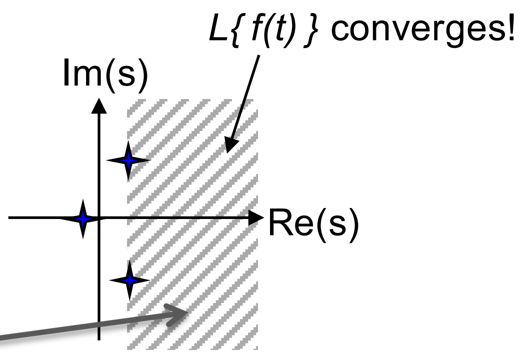
- Region of convergence:
  – Note: For $\mathrm{Re}(s)<0$ the integral may not converge since the factor e^{-st} diverges

  – Consider function $f(t) = e^{\nu t} \Rightarrow F(s) = \int_0^{\infty} e^{(\nu - s)t} dt$

  – $F(s)$ is integrable only if $\mathrm{Re}(s) > \mathrm{Re}(\nu)$

> Thus, the Laplace transform is only valid for
> $\mathrm{Re}(s) > \mathrm{Re}(\nu)$

Note: $f(t) = e^{\nu t}$ means pole at $s=\nu$, i.e.
poles must be to the right of the *region of convergence*



- **Conclusion**: Laplace transform allows studies of unstable modes; $e^{\gamma t}$!

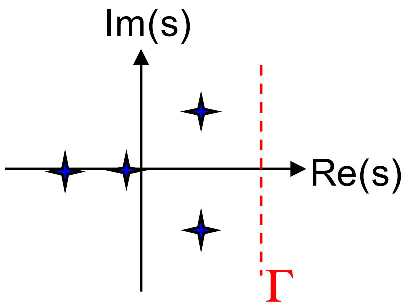
Laplace transform

- Laplace transform

$$F(s) = L\left\{f(t)\right\} = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\infty} e^{-st} f(t) dt$$

- For causal function the inverse transform is:

$$f(t) = L^{-1}\{F(s)\} = \int_{\Gamma - i\infty}^{\Gamma + i\infty} e^{st} F(s) ds$$



  – Here the parameter Γ should be in the ***region of convergence***, i.e. chosen such that all poles lie to the ***right*** of the integral contour Re(s)=Γ.

  – **Causality**: since all poles lie right of integral contour, $L^{-1}\{ f(s) \}(t)=0$, for $t<0$.
    - Proof: see inverse Fourier transform for the causal damped harmonic oscillator
      (Hint: close contour with semicircle Re(s)>0)

  – Thus, **only for causal function** is there an inverse $f(t) = L^{-1}\left\{L\{f(t)\}\right\}$

- Again, Laplace transform allows studies of unstable modes; $e^{\gamma t}$!

Compare Fourier and Laplace transforms

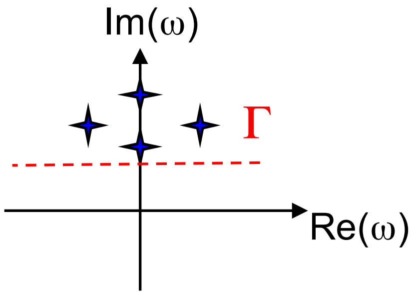
- Formulas for Laplace and Fourier transform very similar
  - Laplace transform for *complex* growth rate s / Fourier for *real* frequencies ω
  - For causal function, Laplace transform is more powerful
  - For causal function, Fourier transforms and Laplace transforms are similar!

- Let $s = i\omega$; provides alternative formulation of the Laplace transform for causal $f(t)$

$$\hat{F}(\omega) = L\{f(t)\} = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{+\infty} e^{-i\omega t} f(t)dt = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)dt$$

- Here ω is a *complex frequency*
- The inverse transform for causal functions is

$$f(t) = L^{-1}\{\hat{F}(\omega)\} = \int_{-i\Gamma-\infty}^{-i\Gamma+\infty} e^{i\omega t} \hat{F}(\omega)dt$$



  - for decaying modes all poles are above the real axis and $\Gamma=0$.
- **Thus, the Laplace and Fourier transforms are the same for amplitude integrable causal function, but only the Laplace transform is defined for exponentially growing functions.**

Summary so far…

- This course is all about waves!
  - The prototycal wave is the *plane wave*: $\exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$
- Sums of waves can be represented using Fourier transforms, but…
  - many important function have no Fourier transform!
  - They can still be transformed as limits of normal function
  - The transform yield *generalised function*, e.g. the Dirac function
- The *Plemej formula*, important for wave damping:

$$\frac{1}{\omega + i0} = \wp \frac{1}{\omega} - i\pi\delta(\omega)$$

- Generalised function allows us to transform plane waves!
- Exponentially growing functions (complex frequencies)
  - Consider only causal functions
  - Use Laplace transform

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Greens functions (Chapter 5)

- **Greens functions**: technique to solve *inhomogeneous* equations
- Linear differential equation for f with source S:

$$L(z)f(z) = S(z)$$

  – where the differential operator L is of the form:

$$L = A_n \frac{d^n}{dz^n} + A_{n-1} \frac{d^{n-1}}{dz^{n-1}} + \dots A_0$$

- Define Greens function G to solve:

$$L(z)G(z,z') = \delta(z-z')$$

  – the response from a point source – e.g. the fields from a particle!

- **Ansatz**: given the Greens function, then there is a solution:

$$f(z) = \int G(z,z')S(z')dz'$$

- **Proof**:

$$L(z)f(z) = \int L(z)G(z,z')S(z')dz' = \int \delta(z-z')S(z')dz' = S(z)$$

How to calculate Greens functions?

- For differential equations without explicit dependence on z, then

$$L(z) = L(z - z')$$

  - we may rewrite G as:

$$G(z,z') \rightarrow G(z - z')$$

- Fourier transform from z-z' to k :

$$L(z - z')G(z - z') = \delta(z - z')$$

$$\Downarrow$$

$$L(ik)G(k) = 1$$

$$\Downarrow$$

$$G(k) = \frac{1}{L(ik)}$$

Example:

$$\left(\frac{\partial^2}{\partial t^2} + \Omega^2\right)G(t - t') = \delta(t - t')$$

$$\Downarrow$$

$$\left(-\omega^2 + \Omega^2\right)G(\omega) = 2\pi$$

$$\Downarrow$$

$$G(\omega) = -\frac{2\pi}{\omega^2 - \Omega^2}$$

- Inverse Fourier transform

$$G(z - z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)e^{-ik(z-z')}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{L(ik)}e^{-ik(z-z')}dk$$

Solve integral!

Greens function for the Poisson's Eq. for static fields

- Poisson's equation

$$-\varepsilon_0 \nabla^2 \phi(\mathbf{x}) = \rho(\mathbf{x})$$

- Green's function $-\varepsilon_0 \nabla^2 G(\mathbf{x}-\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$

$$-\varepsilon_0 \left|\mathbf{k}^2\right| G(\mathbf{k}) = 1$$

$$G(\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^3 \varepsilon_0} \int \frac{\exp\left[i\mathbf{k} \bullet (\mathbf{x}-\mathbf{x}')\right]}{\left|\mathbf{k}^2\right|} d^3\mathbf{k}$$

$$G(\mathbf{x}-\mathbf{x}') = \frac{1}{4\pi\varepsilon_0 \left|\mathbf{x}-\mathbf{x}'\right|}$$

- Thus, we obtain the familiar solution; a sum over all sources

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}$$

Greens Function for d'Alembert's Eq. (time dependent field)

- D'Alembert's Eq. has a Green function $G(t,x)$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(t-t',\mathbf{x}-\mathbf{x}') = \mu_0 \delta(t-t')\delta^3(\mathbf{x}-\mathbf{x}')$$

- Fourier transform $(t-t',\mathbf{x}-\mathbf{x}') \rightarrow (\omega,\mathbf{k})$ gives…

$$\left(\frac{\omega^2}{c^2} - |\mathbf{k}|^2\right) G(\omega,\mathbf{k}) = \mu_0$$

$$G(\omega,\mathbf{k}) = \frac{-\mu_0}{\omega^2/c^2 - |\mathbf{k}|^2}$$

$$G(t-t',\mathbf{x}-\mathbf{x}') = \frac{\mu_0}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta(t-t'-|\mathbf{x}-\mathbf{x}'|/c)$$

  – Information is propagating radially away from the source at the speed of light

Greens Function for the Temporal Gauge

- Temporal gauge gives different form of wave equation

$$\frac{\omega^2}{c^2}\mathbf{A}(\omega,\mathbf{k})+\mathbf{k}\times\mathbf{k}\times\mathbf{A}(\omega,\mathbf{k})=-\mu_0\mathbf{J}(\omega,\mathbf{k})$$

$$\left[\left(\frac{\omega^2}{c^2}-|\mathbf{k}|^2\right)\delta_{ij}+k_ik_j\right]A_j(\omega,\mathbf{k})=-\mu_0 J_i(\omega,\mathbf{k})$$

  – Different response in *longitudinal* : $\mathbf{k}\cdot\mathbf{A}(\omega,\mathbf{k})=-\frac{\mu_0 c^2}{\omega^2}\mathbf{k}\cdot\mathbf{J}(\omega,\mathbf{k})$

  – and *transverse* directions: $\mathbf{k}\times\mathbf{A}(\omega,\mathbf{k})=-\frac{\mu_0}{\omega^2/c^2-|\mathbf{k}|^2}\mathbf{k}\times\mathbf{J}(\omega,\mathbf{k})$

- To separate the longitudinal and transverse parts the Greens function become a 2-tensor G_{ij}

$$\left[\left(\frac{\omega^2}{c^2}-|\mathbf{k}|^2\right)\delta_{ij}+k_ik_j\right]G_{jl}(\omega,\mathbf{k})=-\mu_0\delta_{il}$$

- Solution has poles $\omega = \pm |\mathbf{k}|\, c$:

$$G_{ij}(\omega,\mathbf{k})=-\frac{-\mu_0}{\omega^2/c^2-|\mathbf{k}|^2}\left(\delta_{ij}+\frac{\omega^2}{c^2}k_ik_j\right)$$