ED2210: Assignment 1

Your answers should be handed in 2016-01-26. Discussions between students are allowed, but you may ***not*** copy each others results. The answers may be written down on paper, or electronically. Please, take small step between each line in your derivation to make it easy to understand your algebra. The text has to be easily readable – your credits will be reduced if I cannot read your answers.

1) Magnetic fields can be represented by so called Euler potential $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$.
$$\mathbf{B}(\mathbf{r}) = \nabla\alpha(\mathbf{r})\times\nabla\beta(\mathbf{r})$$
This is also known as the Clebsh representation.
   a) Show that $\mathbf{A}(\mathbf{r}) = \alpha(\mathbf{r})\nabla\beta(\mathbf{r})$ and $\mathbf{A}'(\mathbf{r}) = -\beta(\mathbf{r})\nabla\alpha(\mathbf{r})$ are two possible representations for the vector potential.
   b) Assume $\alpha(\mathbf{r}) = f(x)$ and $\beta(\mathbf{r})$=*g(y)*, where (x, y, z) is a Cartesian coordinate system. What are the directions of the $\mathbf{B}$, $\mathbf{A}$, and $\mathbf{A}'$?
   c) While $\mathbf{A}$, and $\mathbf{A}'$ are in different directions, what do these directions have in common (how do they relate to the direction of $\mathbf{B}$)?
   d) Is this relation between the direction of the vector potential and $\mathbf{B}$ always true for any $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$?
   e) Show that the two choices for the vector potential in problem a) are connected by a gauge transformation. Specifically, identify $\nabla\psi(\mathbf{r})$ in the transformation
$$\mathbf{A}(\mathbf{r}) = \mathbf{A}'(\mathbf{r}) + \nabla\psi(\mathbf{r})$$

2) Use tensor notation to establish the following vector identities:
   a) $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times\nabla\mathbf{B} + \mathbf{B}\times\nabla\mathbf{A}$
   b) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$
   c) $\nabla\times\nabla\phi = 0$

3) The angular momentum density in an electromagnetic field can be defined in terms of the momentum density, $\mathbf{P}_{EM} = \mathbf{E}\times\mathbf{B}\,/\,\mu_0 c^2$ by
$$\mathbf{L}_{EM} = \mathbf{x}\times\mathbf{P}_{EM} = \mathbf{x}\times\frac{(\mathbf{E}\times\mathbf{B})}{\mu_0 c^2}$$
   a) Around which pivotal point does the equation above determine angular momentum?
   b) Show that the continuity equation for angular momentum can be written on the form
$$\frac{\partial}{\partial t}(\mathbf{L}_{EM})_i + \frac{\partial}{\partial x_j}(\mathbf{M}_{EM})_{ij} = (\mathbf{S}_{EM})_i$$
   where
$$(\mathbf{M}_{EM})_{ij} = \epsilon_{irs}(\mathbf{T}_{EM})_{jr}x_s$$
$$(\mathbf{S}_{EM})_i = -\rho\epsilon_{ijk}x_jE_k - J_i(\mathbf{x}\cdot\mathbf{B}) - B_i(\mathbf{x}\cdot\mathbf{J})$$
   In this derivation you may start from the Eq. (3.5) in Melrose & McPhedran:
$$\frac{\partial}{\partial x_j}(\mathbf{T}_{EM})_{ij} = \rho E_i + (\mathbf{J}\times\mathbf{B})_i + \frac{\partial}{\partial t}(\mathbf{P}_{EM})_i$$
   Note: consider carefully the meaning of $\mathbf{x}$.
   c) Considering the vertical component (z-component) of the angular momentum density. What component of the electric field can, at the point where $x = z = 0$ and $y = 1$, produce an angular momentum source? Why this is a reasonable result?