Department of Mathematics
SF2729 Groups and Rings
Period 3, 2016

HOMEWORK 7

Submission. The solutions should be typed and converted to .pdf. Deadline for submission is Monday February 1, 14.00. Either hand in the solutions in class, in the black mailbox for homework outside the math student office at Lindstedtsvägen 25, or by email to `skjelnes@kth.se`.

Score. For each set of homework problems, the maximal score is 3 points. The total score from all twelve homeworks will be divided by four when counted towards the first part of the final exam.

Problem 1. Let V be a $\mathbb{R}$-vector space, and let $E = \mathrm{End}(V)$ denote the set of all linear maps $V \to V$.

(a) Show that E has a natural structure of a vector space, and that E becomes a unitary ring by composition. **(1 p)**
(b) Is E commutative, does it have non-trivial zero divisors? **(1 p)**
(c) If V is of finite dimension, show that E is of finite dimension as well. **(1 p)**

Problem 2. The monomials of order $n \geq 0$ in the polynomial ring $\mathbf{Q}[x, y]$ form a vector space V_n. We have the Poincare series

$$P(t) = \sum_{n \geq 0} \dim(V_n) t^n \quad \text{in} \quad \mathbf{Q}[[t]],$$

where $Q[[t]]$ is the power series ring, see exercise 3, chapter 7.2, page 238. Show that $P(t) = 1/(1-t)^2$. **(3 p)**

Problem 3. For $f \in \mathbb{Q}[x]$ we let $(f) = \{gf \mid g \in \mathbb{Q}[x]\}$. The quotient group $\mathbb{Q}[x]/(f)$ has a natural induced multiplication; $\overline{g} \cdot \overline{h} = \overline{gh}$, where $\overline{p}$ denotes the quivalence class of an element $p \in \mathbb{Q}[x]$.

(a) Show that $A = \mathbb{Q}[x]/(x^2 + 1)$ is a field. **(1 p)**
(b) Multiplication with x determines a $\mathbb{Q}$-linear map $\mu_X \colon A \to A$, where $\mu_x(e) = \overline{x} \cdot e$. Compute the characteristic polynomial of μ_X. **(1 p)**
(c) Determine the characteristic polynomial of the multiplication map $\mu_f \colon A \to A$, for a given $f \in \mathbb{Q}[x]$. **(1 p)**