KTH ROYAL INSTITUTE
OF TECHNOLOGY

ED2210

Lecture 3: The response of a media

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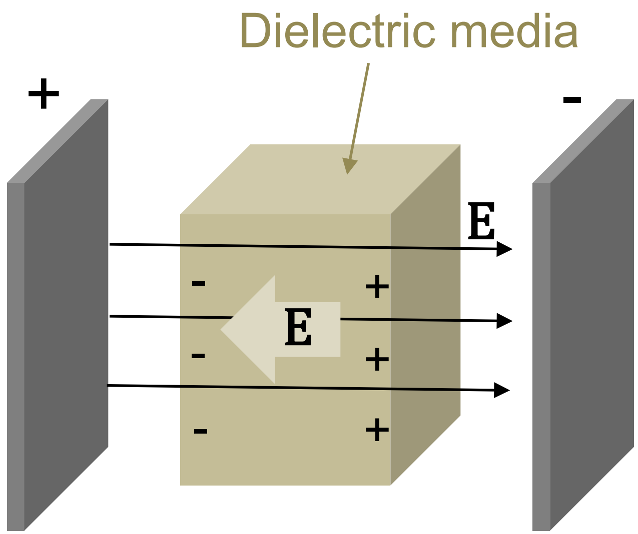
Dielectric media

The theory of dispersive media is derived using the formalism originally developed for dielectrics…let's refresh our memory!

- Capacitors are often filled with a dielectric.
- Applying an electric field, the dielectric gets polarised
- Net field in the dielectric is reduced.



Polarisation

Polarisation is the response of e.g.

- bound electrons, or
- molecules with a dipole moment

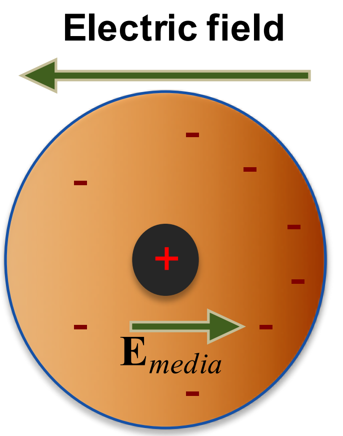
In both cases the particles form dipoles.

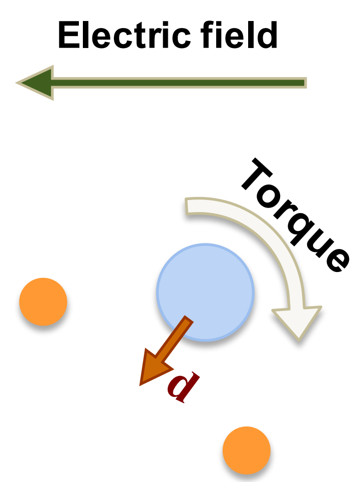
Definition:
The polarisation, $\mathbf{P}$, is the electric dipole moment per unit volume.

Linear media:

$$\mathbf{P} = \chi^e \varepsilon_0 \mathbf{E}$$

where χ^e is the electric susceptibility





Magnetisation

Magnetisation is due to induced dipoles, either from

- induced currents
- quantum mechanically from the particle spin

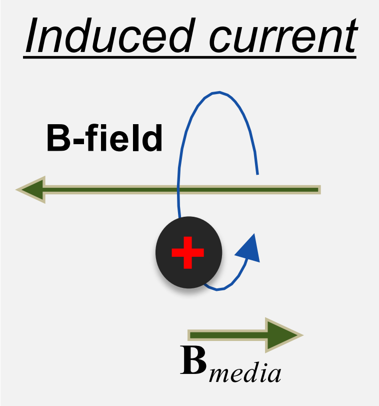
Definition:
The magnetisation, $\mathbf{M}$, is the magnetic dipole moment per unit volume.

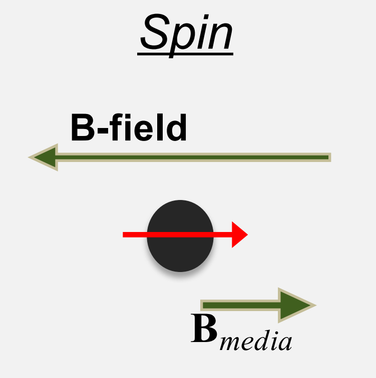
Linear media:

$$\mathbf{M} = \chi^m \mathbf{B}/\mu_0$$

Magnetic susceptibility : χ^m

NOTE: Both polarisation and magnetisation fields are due to dipole fields only; not related to higher order moments like the quadropoles, octopoles etc.





Field calculations with dielectrics

The capacitor problem includes two types of charges:

- Surface charge, ρ_s, on metal surface (monopole charges)
- Induced dipole charges, ρ_{ind}, within the dielectric media

Total electric field strength, $\mathbf{E}$, is generated by $\rho_s + \rho_{ind}$,

The polarisation, $\mathbf{P}$, is generated by ρ_{ind}

The electric induction, $\mathbf{D}$, is generated by ρ_s, where

$$\mathbf{D} := \varepsilon_0 \mathbf{E} + \mathbf{P} = (\mathbf{1} + \chi^e)\varepsilon_0 \mathbf{E} = K\varepsilon_0 \mathbf{E} = \varepsilon \mathbf{E}$$

Here K is the dielectric constant and ε is the permittivity.

The solutions to simple electrostatic problems:

1. Calculate $\mathbf{D}$ from $\nabla \cdot \mathbf{D} = \boldsymbol{\rho_s}$
2. Electric field inside the media is $\mathbf{E} := \mathbf{D}/\varepsilon$

Polarisation physics hidden in ε!

The magnetic field strength

Separating the magnetic fields driven by…

- free electrons, $\mathbf{J}_f$, and
- induced currents in magnetised media, $\mathbf{J}_{ind}$

Define: The magnetic field strength, $\mathbf{H} := \mathbf{B}/\mu_0 - \mathbf{M}$

Linear media may then be characterised by:

- Magnetic susceptibility: χ^m
- Relative permeability: $\mu_r = (1 - \chi^m)^{-1}$
- Permeability: $\mu = \mu_0 \mu_r$

$$\mathbf{H} = \mathbf{B}/\mu = \mathbf{B}/\mu_r\mu_0$$

Multipole expansions and the Fourier transform

The induced charge and current is related to $\mathbf{P}$ and $\mathbf{M}$.

To see this we Fourier transform in space:

$$\rho_{ind}(t,\mathbf{k}) = \int \rho_{ind}(t,\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3\mathbf{x}$$

$$\mathbf{J}_{ind}(t,\mathbf{k}) = \int \mathbf{J}_{ind}(t,\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3\mathbf{x}$$

Expand the exponential for small $|\mathbf{x}|$: $e^{i\mathbf{k}\cdot\mathbf{x}} = \mathbf{1} + i\mathbf{k}\cdot\mathbf{x} + \cdots$

If the system has no net charge, then

Dipole density!

$$\rho_{ind}(t,\mathbf{k}) = \int i\mathbf{k}\cdot[\rho_{ind}(t,\mathbf{x})\mathbf{x} + \cdots]\mathrm{d}^3\mathbf{x} = \int i\mathbf{k}\cdot\mathbf{P}\mathrm{d}^3\mathbf{x} + \cdots$$

$$\mathbf{J}_{ind}(t,\mathbf{k}) = \cdots = \int \left[\frac{\partial \mathbf{P}}{\partial t} + i\mathbf{k}\times\mathbf{M}\right] \mathrm{d}^3\mathbf{x} + \cdots$$

Maxwell's equations for induced charge/current

The inverse Fourier transform (see previous page) provides Maxwell-like equations for $\mathbf{P}$ and $\mathbf{M}$:

$$\nabla \cdot \mathbf{P} = -\rho_{ind}$$

$$\nabla \times \mathbf{M} = \mathbf{J}_{ind} - \frac{\partial \mathbf{P}}{\partial t}$$

If we define the free charge and free current as:

$$\rho_f = \rho - \rho_{ind} , \mathbf{J}_f = \mathbf{J} - \mathbf{J}_{ind}$$

Consequently, Maxwell's Eqs. can be written as:

$$\nabla \cdot \varepsilon \mathbf{E} = \rho_f$$

$$\nabla \times \mu \mathbf{B} = \mathbf{J}_f + \frac{\partial \varepsilon \mathbf{E}}{\partial t}$$

Resistive response

For conducting media, the it is describe the electro-magnetic response with a resistivity, η, or conductivity, σ:

$$\mathbf{J}_{ind} = \sigma \mathbf{E} \, , \mathbf{E} = \eta \mathbf{J}_{ind}$$

From charge continuity

$$\frac{\partial \rho_{ind}}{\partial t} = -\nabla \cdot \mathbf{J}_{ind} = -\nabla \cdot \sigma \mathbf{E}$$

A new formulation of the induced charge and current!

- But ρ_{ind} is only described for time-dependent fields

Is there a relation between σ, ε and μ?

YES!! …part of next weeks homework!

Response tensors

To understand the relation between σ, ε and μ we need to describe the response as a *tensor*.

Let σ_{ij} be the components of the *conductivity tensor*, then

$$J_i = \sigma_{ij} E_j$$

The conductivity tensor allows the current to be in a different direction than the electric field!

Let χ_{ij} and χ_{ij}^{m} be the electric and magnetic susceptibility tensors:

$$P_i = \chi_{ij} E_j \varepsilon_0$$
$$M_i = \chi_{ij}^{m} B_j / \mu_0$$

In this course all responses are tensors

Finally, we have the dielectric tensor, K_{ij}: $D_i = K_{ij} E_j \varepsilon_0$

Uniqueness of response tensors

- The response tensors (previous page) allows us to describe any linear response of a media!
- In fact, response tensors are so powerful that if we know one tensor, then we can calculate all others!
- The relation can be simply formulated only in Fourier space (for plane waves):

$$K_{ij}(\omega,\mathbf{k}) = \delta_{ij} + \frac{i}{\omega\varepsilon_0}\sigma_{ij}(\omega,\mathbf{k})$$
$$= \delta_{ij} + \chi_{ij}(\omega,\mathbf{k})$$
$$= \delta_{ij} + \frac{1}{\omega^2\varepsilon_0}\alpha_{ij}(\omega,\mathbf{k})$$

Polarisation tensor α_{ij}:
$$J_i = \alpha_{ij}A_j$$

For the relation to the magnetic suseptibility, see Melrose and McPhedran.

What are response tensors like?

The simplest type of conductivity is scalar, e.g. $J_i = \sigma E_i$
How are they represented using tensors?

Use the identity: $A_i = \delta_{ij} A_j$ (where δ_{ij} is the unit matrix!)
Thus: $\sigma_{ij} = \sigma \delta_{ij} \rightarrow J_i = \sigma_{ij} E_j = \sigma E_i$

The corresponding scalar dielectric constant, K:

$$K\delta_{ij} = K_{ij} = \delta_{ij} + \frac{i}{\omega\varepsilon_0}\sigma_{ij} = \delta_{ij}\left(1 + \frac{i}{\omega\varepsilon_0}\sigma\right)$$

$$K = 1 + \frac{i}{\omega\varepsilon_0}\sigma$$

What are response tensors like?

What is the physics behind…

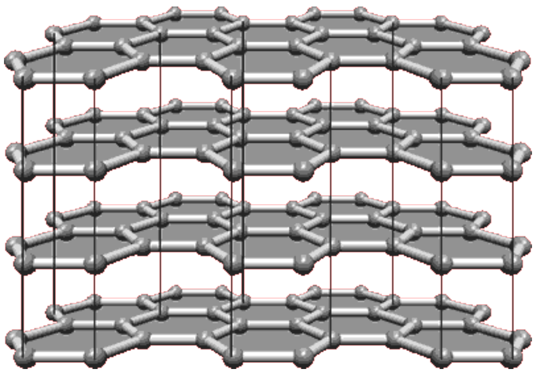
$$\sigma_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

What is the current when $E_i = [1 \quad 0 \quad 0]$ and $E_i = [0 \quad 0 \quad 1]$?

Note: Different response in different direction!

Such medias are said to be **anisotropic**.

$\hat{n}$



Crystal structure of graphite

Example:

Graphite is built from layers of strong bounds. Its conductivity along the layers is ~ 1000 times higher than across the layers.

Gyrotropic media

When there is a strong static field through a media, then it may become *gyrotropic*

$$\sigma_{ij} = \sigma\delta_{ij} + \tau\epsilon_{ijk}B_k$$

What are the components of σ_{ij} for $\mathbf{B}$ is in the z-direction?

- Gyrotropic medias are also anisotropic since they have different response in different directions
- Magnetised plasmas are often, but not always, gyrotropic.

Kerr effect and Cotton-Moulton effect

The Kerr effect: is caused by an electro-static field, $\mathbf{E}^0$:

$$K_{ij} = K^0\delta_{ij} + \beta E_i^0 E_j^0$$

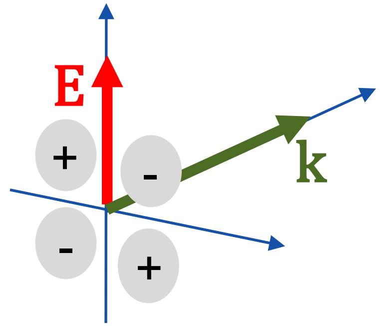
The Cotton-Moulton effect is caused by a static magnetic field, $\mathbf{B}^0$:

$$K_{ij} = K^0\delta_{ij} + \gamma B_i^0 B_j^0$$

Optically active media

Some media have a strong quadro-pole response.

- Such media are called *chiral* media
  - non mirror symmetric
- The response is said to be *optically active*.
- Linearly polarised waves tend to rotated!

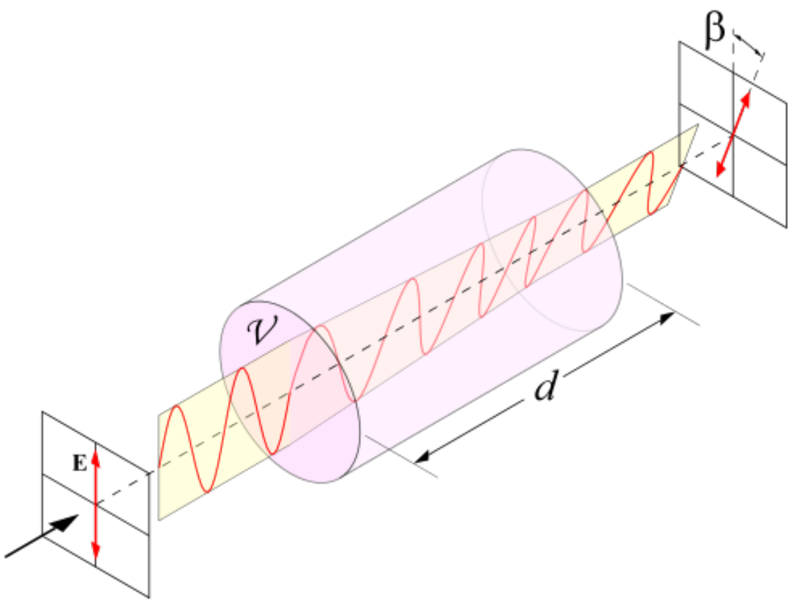


The typical dielectric tensor is:

$$K_{ij} = a\delta_{ij} + b\epsilon_{ijm}k_m$$

Note: This tensor is isotropic.

There is no intrinsic direction in the media and the response is rotationally symmetric.



Author: DrBob~commonswiki

Time dependent responses

Particles have mass/inertia, thus accelerated by an electric field it takes takes time.

The total flow of charged (current) depends on the time-history. For a single particle:

$$v_i(t) = v_i(-\infty) + \int_{-\infty}^{t} \frac{q}{m} E_i(t')dt'$$

In general, the current is a convolution in both time and space

$$J_i(\mathbf{x}, t) = \int d^3x' \int_{-\infty}^{t} dt' \; \sigma_{ij}(\mathbf{x} - \mathbf{x}', t - t') E_i(\mathbf{x} - \mathbf{x}', t')$$

Thus, this representation allows:

- Delayed responses, from earlier acceleration
- Non-local responses; the current in $\mathbf{x}$ depends on the electric field in $\mathbf{x} - \mathbf{x}'$.

Dispersive media

Medias are called dispersive if the dielectric tensor depends on either ω or $\mathbf{k}$.
Except for terms proportional to $(k_i k_j - k^2 \delta_{ij})/\omega^2$.

Temporal dispersion: $\mathbf{K}(\omega)$

Spatial dispersion: $\mathbf{K}(\mathbf{k})$

Why do we have dispersion?

- Medias with a natural frequency are temporally dispersion.
- Medias with a characteristic wave length are spatially dispersion.
- Characteristic velocities cause a mix of temporal and spatial dispersion.

Can you think of any examples?

Delayed response

Inertia (mass) tend to delay the electromagnetic response, as it takes time to accelerate particles. This causes dispersion!

Example:

- The time scale for free electron to respond is the inverse plasma frequency
- Only for frequencies comparable, or lower than the plasma frequency do free electrons contribute to the response.
- The response depends on the frequency – *temporal dispersion*!

Non-local response

Consider a long molecule.

If you push it at one end, the whole molecule will start moving!

If the molecule is much longer than the wave length, then the total "push" from the wave averages out – the response is zero/small!

If the wave length is much longer than the molecule, then the molecule may respond as if it was a point charge.

Thus the response depends on $\mathbf{k}$ – spatial dispersion!

Nonlinear media

Some media are non-linear, e.g.

$$P_i(\omega) = \chi_{ij}(\omega)E_j(\omega) + \int d\omega_1 d\omega_2 \chi_{ijk}(\omega,\omega_1,\omega_2)E_j(\omega_1)E_k(\omega_2) +$$

$$+ \int d\omega_1 d\omega_2 d\omega_3 \chi_{ijkm}(\omega,\omega_1,\omega_2,\omega_3)E_j(\omega_1)E_k(\omega_2)E_m(\omega_3) + \cdots$$

The non-linear terms are that they produce new frequencies and waves lengths.

- Assume that $E \sim e^{-i\omega t}$ …
- …then the quadratic term above produces a current: $J \sim e^{-i2\omega t}$
- …and the cubic term will produce a current: $J \sim e^{-i3\omega t}$

In general multiplying two plane wave (ω_1, k_1) and (ω_2, k_2) drives a response: $(\omega_1 + \omega_2, k_1 + k_2)$

Exercise 1.2: Time-averaged work

Consider an electric field

$$\mathbf{E} = \mathbf{e}_x \Re\{E\} \quad , \quad E = E_0 \mathrm{e}^{\mathrm{i}\phi} \quad , \quad \phi = kx - \omega t$$

and a current

$$\mathbf{J} = \mathbf{e}_x \Re\{J\} \quad , \quad J = J_0 \mathrm{e}^{\mathrm{i}\phi + i\delta}$$

where E_0 and J_0 are real amplitudes and the phase difference δ is real.

a) What is the work performed by the electric field on the charged particles that carry the current?

b) What is the time average of this work?

c) What is the phase difference, δ, that minimizes and maximizes the time averaged work, and what δ gives zero time-averaged work?

d) Also at these minima and maxima, when is the energy transferred from the wave to the particles (absorption) and vice versa (emission)?

Exercise 1.3: Hermitian and anti-hermitian tensors

In 2 dimensions, what number of real parameters is needed to describe…

a)any hermitian 2-tensor?

b)any anti-hermitian 2-tensor?

Note: if T^H is the hermitian nd T^A the antihermitian parts of T then:

$$T^H = \frac{1}{2}\left(T + T^\dagger\right)$$

$$T^A = \frac{1}{2}\left(T - T^\dagger\right)$$

$$T^\dagger = transpose\left(T^*\right)$$

Exercise 1.4: Time-averaging and Fourier representations

a) Express the time-averaged energy density of an electric field $E(t)$ using a Fourier representation. Here the time-averaging should be performed over all times t in $(-\infty,\infty)$.

b) Use this expression to evaluate the energy density of the field

$$E(t) = E_0 \exp\left[-i\omega_m t\right]$$

Hint: Use Parseval's theorem:

$$\int_{-\infty}^{\infty} \left| f(t)^2 \right| dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{f}(\omega)^2 \right| d\omega$$

$$\hat{f}(\omega) = \mathbf{F}\left\{ f(t) \right\}$$

Exercise 1.5: Conductivity and work

Consider a current $\mathbf{J}(\mathbf{x},t)$ driven linearly by the electric field $\mathbf{E}(\mathbf{x},t)$ such

$$\mathbf{J}(\mathbf{k},\omega) = \sigma \bullet \mathbf{E}(\mathbf{k},\omega)$$

where σ is the conductivity tensor (2-tensor, i.e. has matrix representations with components σ_{ij}).

How does the work performed by electric field $\mathbf{E}$ on a current $\mathbf{J}$ depend on the hermitian and anti-hermitian parts of the conductivity tensor?

Hint: Use Plancherel's theorem

$$\int_{-\infty}^{\infty} f(t) g^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}^*(\omega) \hat{g}(\omega) d\omega$$

Here g^* is the complex conjugate of g.