Response tensors of ideal media

T. Johnson

Questions

- What is meant by the electromagnetic response of a media?
  - This lecture we will see how to calculate a response tensor.
- What is meant by temporal dispersion, spatial dispersion and anisotropy? Under what conditions do these three effects appear?
  - This lecture we'll calculate the response of dispersive and anisotropic media.

Overview

- Introduction to the concept of a response
- First example: Response of electron gas
  - Changing the speed of light and dispersion
- Polarization of atoms and molecules (brief)
- Properties/symmetries of response tensors
- Medium of oscillators
  - Detailed study of the resonance region
    - Hermitian / antihermitian parts of the dielectric tensor
    - Application of the Plemej formula
- Dielectric response for plasmas
  - Magnetoionic theory (anisotropic/gyrotropic)
  - Cold plasmas (Alfven velocity)
  - Warm plasmas (Landau damping)

What do we mean by dielectric response?

- When an electromagnetic wave passes through a media, e.g. air, water, copper, a crystal or a plasma, then:
  - The electromagnetic fields exert a force on the particles of the media
  - The force may then "pull" the particles to induce
    - charge separation ρ ⇨ drive $\mathbf{E}$-field in Poisson's equation

$$\nabla \bullet \mathbf{E}_{media} = \rho_{media} / \varepsilon_0$$

  - $\mathbf{E}$-field is coupled to the $\mathbf{B}$-field through Maxwells equations
    - currents $\mathbf{J}$ ⇨ drive $\mathbf{E}$- & $\mathbf{B}$-fields through Ampere's law

$$\nabla \times \mathbf{B}_{media} - \frac{1}{c^2} \frac{\partial \mathbf{E}_{media}}{\partial t} = \mu_0 \mathbf{J}_{media}$$

  - The fields induced by the media are called the *dielectric response*
  - The total fields are:

$$\mathbf{E} = \mathbf{E}_{external} + \mathbf{E}_{media}$$

$$\mathbf{B} = \mathbf{B}_{external} + \mathbf{B}_{media}$$

See previous lecture for representation in terms of:
- Polarization P
- Magnetization M

Equations for calculating the dielectric response

E- & B-field exerts a force on the particles in media

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \textit{solve for } \mathbf{v}\ !$$

The induced motion of charge particles form a current and a charge density

$$\mathbf{J}_{media} = \sum_{species} qn\mathbf{v} \qquad \frac{\partial}{\partial t}\rho_{media} + \nabla \circ \mathbf{J}_{media} = 0$$

(n=particle density)

The **response** can be quantified by the conductivity σ

$$J_i(\mathbf{k},\omega) = \sigma_{ij}(\mathbf{k},\omega) E_j(\mathbf{k},\omega)$$

Current and charge drive the **electromagnetic response**

$$\nabla \bullet \mathbf{E}_{media} = \rho_{media} / \varepsilon_0$$

$$\nabla \times \mathbf{B}_{media} - \frac{1}{c^2}\frac{\partial \mathbf{E}_{media}}{\partial t} = \mu_0 \mathbf{J}_{media}$$

Response of *electron gas* to oscillating E-field

Example: Consider electron response to electric field oscillations (e.g. high frequency, long wave length waves in a plasma)

•Align x-axis with the electric field: $\mathbf{E}(t) = \mathbf{e}_x E_x(t)$

•Electron equation of motion:

$$m\ddot{x}(t) = qE_x(t) \quad \Rightarrow \quad x(\omega) = -\frac{q}{m\omega^2}E_x(\omega)$$

•The current driven in the medium (let n be the electron density)

$$J_x(t) \equiv qn\dot{x}(t) \quad \Rightarrow \quad J_x(\omega) = i\frac{q^2 n}{m\omega}E_x(\omega)$$

•Thus we have derived the conductivity of this media

$$\boxed{\sigma(\omega) = i\frac{nq^2}{m\omega}}$$

•Here: $\sigma \sim 1/\omega$, means that the media is *dispersive*!

Response of *electron gas* to oscillating E-field (2)

- This media is ***isotropic*** (the same response in all directions)
  - **Proof 1**: rotate E-field to align with y-axis or z-axis and repeat calculation
  - **Proof 2**: use argument that the medium have no "intrinsic direction" (there is no static the magnetic field, no structure like in a crystal, or similar), thus the media have to be isotropic
  - Being an isotropic media the components of the conductivity tensor are:

$$\sigma_{ij}(\omega) = \sigma(\omega)\delta_{ij} \quad \Rightarrow \quad \sigma_{ij}(\omega) = i\frac{q^2 n}{m\omega}\delta_{ij} \equiv i\varepsilon_0 \frac{\omega_p^2}{\omega}\delta_{ij}$$

where ω_p is known as the *plasma frequency*: $\omega_p^2 \equiv \dfrac{nq^2}{\varepsilon_0 m}$

- Other response tensors:
  - susceptibility: $$\chi_{ij}(\omega) \equiv \frac{i}{\varepsilon_0\omega}\sigma_{ij}(\omega) = \frac{i\sigma(\omega)}{\varepsilon_0\omega}\delta_{ij} = -\frac{\omega_p^2}{\omega^2}\delta_{ij}$$
  - polarisation response: $$\alpha_{ij}(\omega) \equiv i\omega\sigma_{ij}(\omega) = -\varepsilon_0\omega_p^2\delta_{ij}$$
  - dielectric tensor: $$K_{ij}(\omega) \equiv \delta_{ij} + \chi_{ij}(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\delta_{ij}$$

Application of response

- How does the electron response affect the propagation of waves?
  - Consider: high frequency, long wave length waves in a plasma
    - then response tensor from previous page is valid (more details later)
- Split currents into antenna current J_{ant} and the current induced in the media J_{media}. Then Amperes and Faradays equations give:

$$\mathbf{k}\times\mathbf{k}\times\mathbf{E}+\frac{\omega^2}{c^2}\mathbf{E}+i\mu_0\omega\mathbf{J}_{media}=-i\mu_0\omega\mathbf{J}_{ant}$$

Note: total field E driven by both J_{media} and J_{ant}

- Use the conductivity $\sigma=i\varepsilon_0\frac{\omega_p^2}{\omega}$ of the media:

$$\frac{\omega^2}{c^2}\mathbf{E}+i\mu_0\omega\mathbf{J}_{media}=\frac{\omega^2}{c^2}\mathbf{E}+i\mu_0\omega\sigma\mathbf{E}=\omega^2\underbrace{\frac{1}{c^2}\left(1-\frac{\omega_p^2}{\omega^2}\right)}_{1/c_m^2}\mathbf{E}$$

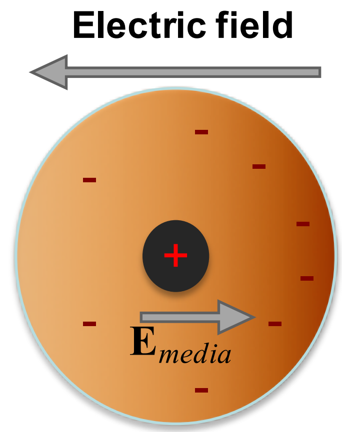
$$\Rightarrow\quad\mathbf{k}\times\mathbf{k}\times\mathbf{E}+\frac{\omega^2}{c_m^2}\mathbf{E}=-i\mu_0\omega\mathbf{J}_{ant}$$

i.e. a wave equation with speed of light: $c_m^2=c^2\left(1-\frac{\omega_p^2}{\omega^2}\right)^{-1}$

Polarization of atoms and molecules

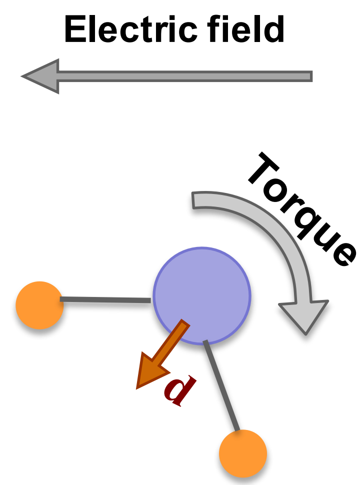
- The polarization of an atom (requires quantum mechanics)
  - The electric field pushes the electrons, inducing a charge separation;
  - Quantum mechanically: perturbs the eigenfunctions (orbitals): $\psi^{(0)} \rightarrow \psi^{(0)}+\psi^{(1)}$

$$\psi_q^{(1)} = \sum a_{qq'} \psi_{q'}^{(0)} \rightarrow J_{media}^{(1)} \;\&\; \rho_{media}^{(1)}$$



  - The field induced by the media is opposite to the total field

- The polarization of a water molecule
  - Water molecules, dipole moment **d**
  - The electric field induces a *torque* that turns it to reduce the total field
  - *Note*: the electron eigenstates of the molecules are also perturbed, like in the atom



Uniaxial crystals

- In solids the response, or electron mobility, is determined by the
  - **Metals**: the valence electron give rapid response
  - **Insulators**: electrons orbitals are bound to a single atom or molecule
- Uniaxial crystals: have an optical axis; e.g. the normal $\hat{n}$ to a sheeth structure
- Stronger bonds within then between the sheeths
  - Graphite: valence electrons are shared only within a sheeth
  - electron ***mobility*** (response) is different within and perpendicular to the sheeths
  - The crystal is ***anisotropic***
- Let the normal to the crystal be in the z-direction (as in figure)

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_\perp & 0 & 0 \\ 0 & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{bmatrix}$$

Graphite

$$\begin{cases} \sigma_\parallel = 2.5 - 5.0 \times 10^{-6} \\ \sigma_\perp = 3 \times 10^{-3} \end{cases}$$

- *Example*: slight birefrigence in optical fibres can cause *modal dispersion*

Biaxial crystals

- Uniaxial crystals has symmetric plane, in which the electron mobility is constant
- Biaxial crystals have no symmetry plane
  - Instead they have different conductivity in all three directions

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_\alpha & 0 & 0 \\ 0 & \sigma_\beta & 0 \\ 0 & 0 & \sigma_\gamma \end{bmatrix}$$

  - When expressed in terms of the dielectric tensor one may introduce three refractive indexes of the media

$$\left[K_{ij}\right] = \left[\delta_{ij} + \frac{i}{\omega\varepsilon_0}\sigma_{ij}\right] = \begin{bmatrix} (n_\alpha)^2 & 0 & 0 \\ 0 & (n_\beta)^2 & 0 \\ 0 & 0 & (n_\gamma)^2 \end{bmatrix}$$

Epsom Salt ($MgSO_4$):
n_j = [1.433, 1.455, 1.461]

These medias are rarely strongly unisotropic

Overview

- Introduction to the concept of a response
- First example: Response of electron gas
  - Changing the speed of light and dispersion
- Polarization of atoms and molecules (brief)
- Properties/symmetries of response tensors
- Medium of oscillators
  - Detailed study of the resonance region
    - Hermitian / antihermitian parts of the dielectric tensor
    - Application of the Plemej formula
- Dielectric response for plasmas
  - Magnetoionic theory (anisotropic/gyrotropic)
  - Cold plasmas (Alfven velocity)
  - Warm plasmas (Landau damping)

Anithermitian part of dielectric tensor

- Next lecture we'll see that:
  - Hermitian part of K determines the refractive index, $K^H \sim n^2$
  - Antihermitian part of K determines the damping rate, $K^A \propto \gamma$
- How does K^A relate to damping? Study the work by electric fields:

$$W = \int d^3x dt\, E_i(t,\mathbf{x}) J_j(t,\mathbf{x}) = \int d^3k d\omega\, E_i(\omega,\mathbf{k}) J_i^*(\omega,\mathbf{k})$$

Plancherel's theorem

- Dissipative work: $Re\{W\} = \frac{1}{2}(W + W^*) =$

$$= \frac{1}{2}\int d^3k d\omega \left(E_i \sigma_{ij}^* E_j^* + E_i^* \sigma_{ij} E_j\right) = \int d^3k d\omega\, E_i^* \sigma_{ij}^H E_j$$

- How is the antihermitian part K^A related to the hermitian part σ^H?

$$K_{ij}^A = \frac{1}{2}\left\{\delta_{ij} + \frac{i}{\varepsilon_0 \omega}\sigma_{ij} - \left(\delta_{ji} + \frac{i}{\varepsilon_0 \omega}\sigma_{ji}\right)^*\right\} = \frac{1}{2}\left\{\frac{i}{\varepsilon_0 \omega}\sigma_{ij} - \frac{-i}{\varepsilon_0 \omega}\sigma_{ji}^*\right\} = \frac{i}{\varepsilon_0 \omega}\sigma_{ji}^H$$

Damping caused by: hermitian σ, or antihermitian K!

Positive and negative Frequencies

- Consider a plane wave representation real space and time:

$$E(x,t) = Re\{\hat{E}e^{ikx-i\omega t}\} = \frac{1}{2}\left(\hat{E}e^{ikx-i\omega t} + \hat{E}^*e^{-ikx+i\omega t}\right)$$

$$D(x,t) = Re\{\hat{D}e^{ikx-i\omega t}\} = \frac{1}{2}\left(\hat{D}e^{ikx-i\omega t} + \hat{D}^*e^{-ikx+i\omega t}\right) \quad \text{(A)}$$

- Thus, to represent a wave in space and time we need in fact two plane waves with opposite frequencies and wave number.
  - Two complex waves are needed for a single real-space wave…
  - …that means, their dielectric response have to be related!
- The amplitude of $\hat{E}$ and $\hat{D}$ are related via the dielectric tensor

$$\hat{D}(\omega,k) = \varepsilon_0 K_{ij}(\omega,k)\hat{E}(\omega,k)$$

$$D(x,t) = \frac{1}{2}\left(K_{ij}(\omega,k)\hat{E}e^{ikx-i\omega t} + K_{ij}(-\omega,-k)^*\hat{E}^*e^{-ikx+i\omega t}\right) \quad \text{(B)}$$

- Comparing (A) and (B) we get:

$$K_{ij}(\omega,k) = K_{ij}(-\omega,-k)^*$$

Kramer-Kroniger relations

- Inverse Fourier transformation of relation between $\mathbf{E}$ and $\mathbf{D}$:

$$D_i(\omega,\mathbf{k}) = \varepsilon_0 K_{ij}(\omega,\mathbf{k}) E_i(\omega,\mathbf{k})$$

$$D_i(t,\mathbf{r}) = \varepsilon_0 \int dt' \int d^3x\, K_{ij}(t-t',\mathbf{r}-\mathbf{r}') E_i(t',\mathbf{r}')$$

- Thus, since $D_i(t,r)$ and $E_i(t,\mathbf{r})$ is real, also $K_{ij}(t,\mathbf{r})$ must be real!
  - Fourier transform of real functions gives symmetry:

    $K_{ij}(\omega,k) = K_{ij}(-\omega,-k)^*$ (as on previous page)

- **Causality**: only history impact the future:

  $K_{ij}(t-t',\mathbf{r}-\mathbf{r}') = 0$ for $t' > t$

- For causal function: $f(t) = H(t)f(t)$
  - After a few lines of algebra…the Kramer-Kronig relations:

$$K_{ij}^H(\omega,\mathbf{k}) - \delta_{ij} = \frac{i}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{K_{ij}^A(\omega,\mathbf{k})}{\omega-\omega'}$$

$$K_{ij}^A(\omega,\mathbf{k}) = \frac{i}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{K_{ij}^H(\omega,\mathbf{k}) - \delta_{ij}}{\omega-\omega'}$$

Time reversal

- The dielectric response is a mechanical response due to electromagnetic perturbations.
  - Thus, any realistic response has to be consistent with the laws of mechanics (Newton or Schrödinger)

- Consider Newton's equation of motion

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

- Lars Onsager found that there is an important symmetry w.r.t time reversal (in both Newton's and Schrödinger's equation)

$$t \to -t\ , \mathbf{p} \to -\mathbf{p}, \mathbf{B} \to -\mathbf{B}$$

Onsager relation

- Reactive and resistive responses:

$$\mathbf{J}(\omega,\mathbf{k}) = \sigma\,(\omega,\mathbf{k})\cdot\mathbf{E}(\omega,\mathbf{k}) = (\sigma^{H}\,(\omega,\mathbf{k}) + \sigma^{H}\,(\omega,\mathbf{k}))\cdot\mathbf{E}(\omega,\mathbf{k})$$

- The current can be split in reactive and resistive parts:

$$\mathbf{J}^{resist} = \sigma^{H}\cdot\mathbf{E}$$
$$\mathbf{J}^{react} = \sigma^{A}\cdot\mathbf{E}$$

- Next: Let a particle be accelerated by a wave. If time is reversed…
  - the work on *reactive current* should be transferred back to the wave

$$\mathbf{E}(\omega,\mathbf{k})\cdot\mathbf{J}_{B}^{react}(\omega,\mathbf{k}) = -\mathbf{E}(-\omega,\mathbf{k})\cdot\mathbf{J}_{-B}^{react}(-\omega,\mathbf{k})$$
$$\rightarrow \sigma_{B}^{A}(\omega,\mathbf{k}) = -\sigma_{-B}^{A}(-\omega,\mathbf{k})$$

  - the work on *resistive current* should be unchanged

$$\mathbf{E}(\omega,\mathbf{k})\cdot\mathbf{J}_{B}^{resist}(\omega,\mathbf{k}) = \mathbf{E}(-\omega,\mathbf{k})\cdot\mathbf{J}_{-B}^{resist}(-\omega,\mathbf{k})$$
$$\rightarrow \sigma_{B}^{A}(\omega,\mathbf{k}) = \sigma_{-B}^{A}(-\omega,\mathbf{k})$$

- Combine with reality condition, $\sigma_{ij}\,(\omega,\mathbf{k}) = \sigma_{ji}\,(-\omega,-\mathbf{k})$:

$$\sigma_{ij,B}\,(\omega,\mathbf{k}) = \sigma_{ji,-B}\,(\omega,-\mathbf{k})$$
$$K_{ij,B}\,(\omega,\mathbf{k}) = K_{ji,-B}\,(\omega,-\mathbf{k})$$

These are the *Onsager relations*!

Onsager's relations: Magnetised media

- Consider a magnetised media with $\mathbf{B} = B\mathbf{e}_z$ along the z-axis

$$K_{ij}\left(\omega, \mathbf{k}\right) = K\delta_{ij,B} + L\epsilon_{ijk}B_k = \begin{bmatrix} K & LB & 0 \\ -LB & K & 0 \\ 0 & 0 & K \end{bmatrix}$$

- The transpose of this matrix changes the signs of the off-diagonal element.
- The map $\mathbf{B} \rightarrow -\mathbf{B}$ has the same effect
- Thus, leaving the matrix unchanged as predicted by Onsager!

Overview

- Introduction to the concept of a response
- First example: Response of electron gas
  - Changing the speed of light and dispersion
- Polarization of atoms and molecules (brief)
- Properties/symmetries of response tensors
- Medium of oscillators
  - Detailed study of the resonance region
    - Hermitian / antihermitian parts of the dielectric tensor
    - Application of the Plemej formula
- Dielectric response for plasmas
  - Magnetoionic theory (anisotropic/gyrotropic)
  - Cold plasmas (Alfven velocity)
  - Warm plasmas (Landau damping)

Reminder: Equations for calculating the dielectric response

E- & B-field exerts force on particles in media

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \textit{solve for } \mathbf{v}\ !$$

The induced motion of charge particles form a current and a charge density

$$\mathbf{J}_{media} = \sum_{species} qn\mathbf{v} \qquad \frac{\partial}{\partial t}\rho_{media} + \nabla \circ \mathbf{J}_{media} = 0$$

(n=particle density)

The response can be quantified in e.g. the conductivity σ

$$J_i(\mathbf{k},\omega) = \sigma_{ij}(\mathbf{k},\omega) E_j(\mathbf{k},\omega)$$

Current and charge drive the **dielectric response**

$$\nabla \bullet \mathbf{E}_{media} = \rho_{media} / \varepsilon_0$$

$$\nabla \times \mathbf{B}_{media} - \frac{1}{c^2}\frac{\partial \mathbf{E}_{media}}{\partial t} = \mu_0 \mathbf{J}_{media}$$

Medium of oscillators – dispersive media

- Consider a medium consisting of charged particles with
  - charge q , mass m , density n
- Let the particles position x follow the equation of a forced oscillator
  - i.e. the media has an eigenfrequency Ω and a damping rate Γ
    - damping could be due to collisions (resistivity) and the eigenfrequency could be due to magnetization an acustic eigenfrequency of a crystal

$$\ddot{x}(t) + \Gamma \dot{x}(t) + \Omega^2 x(t) = \frac{q}{m} E_x(t) \quad \Longrightarrow \quad x(\omega) = \frac{q/m}{\Omega^2 - \omega^2 - i\Gamma\omega} E_x(\omega)$$

- The current is then

$$J(\omega) = qn\left[-i\omega x(\omega)\right] = -\frac{i\omega n q^2/m}{\Omega^2 - \omega^2 - i\Gamma\omega} E_x(\omega) \equiv \sigma E_x(\omega)$$

- Thus the dielectric tensor reads

$$K_{ij} = \delta_{ij} + \frac{i}{\varepsilon_0 \omega}\sigma_{ij} = \left(1 + \frac{\omega_p^2}{\Omega^2 - \omega^2 - i\Gamma\omega}\right)\delta_{ij} \quad , \quad \text{where} \quad \omega_p^2 \equiv \frac{nq^2}{\varepsilon_0 m}$$

  - again ω_p is the plasma frequency

Medium of oscillators (2)

- Isotropic dielectric tensors K_{ij} can be replaced by a scalar K, consider e.g. the inner product $K_{ij}E_j = K\delta_{ij}E_j = KE_i$
- For the medium of harmonic oscillators

$$K = 1 + \frac{\omega_p^2}{\Omega^2 - \omega^2 - i\Gamma\omega}$$

- In the *high frequency* limit where $\omega >> \Omega$ and $\omega >> \Gamma$, then

$$K = 1 - \frac{\omega_p^2}{\omega^2} + \ldots$$

  – this is the response of the electron gas!
- At *low frequency* $\omega << \Omega$ and $\omega \sim \Gamma$, then

$$K = 1 + \frac{\omega_p^2}{\Omega^2}$$

  – here the medium is no longer dispersive (independent of ω)

Medium of oscillators (3)

- The medium has "*strongly dispersive*" when the frequency is near the characteristic frequency of the medium $\omega \sim \Omega$
  - To see this, first rewrite the denominator

$$D \equiv \Omega^2 - \omega^2 - i\Gamma\omega =$$

$$= \Omega^2 - (\omega + i\Gamma/2)^2 - \Gamma^2/4$$

$$= \left(\Omega - \omega - i\Gamma/2\right)\left(\Omega + \omega + i\Gamma/2\right) - \Gamma^2/4$$

  - assume here the damping rate to be small $\omega >> \Gamma$ such that the last last term is negligible
  - Next use the relation: $\dfrac{1}{(a-b)(a+b)} = \dfrac{1}{2b}\left(\dfrac{1}{a-b} - \dfrac{1}{a+b}\right)$
  - The dielectric constant is then

$$K \approx 1 - \frac{\omega_p^2}{(\omega + i\Gamma/2 - \Omega)(\omega + i\Gamma/2 + \Omega)}$$

$$= 1 - \frac{\omega_p^2}{2\Omega}\left[\frac{1}{\omega + i\Gamma/2 - \Omega} - \frac{1}{\omega + i\Gamma/2 + \Omega}\right]$$

Medium of oscillators (4)

- Next we shall use the condition that we are close to resonance; i.e. the frequency is near the characteristic frequency $\omega \sim \Omega$:

$$|\omega - \Omega| << |\omega + \Omega| \quad \Rightarrow \quad \left|\frac{1}{\omega - \Omega + i\Gamma/2}\right| >> \left|\frac{1}{\omega + \Omega + i\Gamma/2}\right|$$

- The dielectric constant then reads

$$K \approx 1 - \frac{\omega_p^2}{2\Omega} \frac{1}{(\omega - \Omega + i\Gamma/2)} = 1 - \frac{\omega_p^2}{2\Omega} \frac{(\omega - \Omega - i\Gamma/2)}{\left[(\omega - \Omega)^2 + \Gamma^2/4\right]}$$

$$\begin{cases} K^H \equiv \Re\{K\} = 1 - \dfrac{\omega_p^2}{2\Omega} \dfrac{\omega - \Omega}{\left[(\omega - \Omega)^2 + \Gamma^2/4\right]} \\ K^A \equiv \Im\{K\} = \dfrac{\omega_p^2}{\Omega} \dfrac{\Gamma}{\left[(\omega - \Omega)^2 + \Gamma^2/4\right]} \end{cases}$$

Hermitian: wave propagation (reactive response)

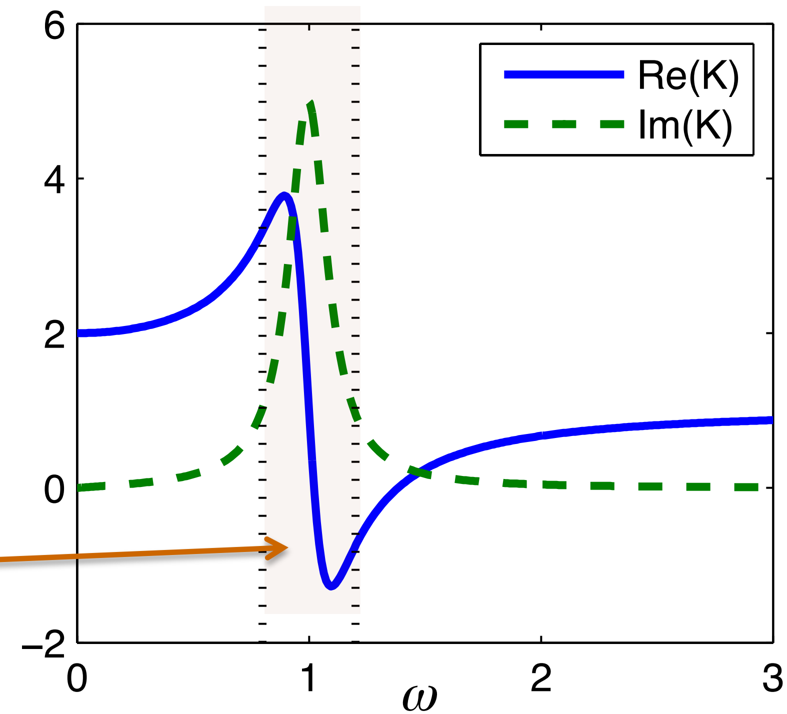
Antihermitian: wave absorption (resistive response)

Medium of oscillators (5)

- Antihermitian part comes from $i\Gamma/2$ in

$$K \approx 1 - \frac{\omega_p^2}{2\Omega} \frac{1}{(\omega - \Omega + i\Gamma/2)}$$

  - which is most important if $|\Gamma/2| \sim |\omega - \Omega|$ (for $|\Gamma/2| << |\omega - \Omega|$ then $K^A << K^H$)
  - Thus, the dissipation occur mainly where $|\Gamma| > |\omega - \Omega|$



- Summary:
  - Low frequency: not dispersive
  - Resonant region: strong damping in ***thin layer*** $|\Gamma| > |\omega - \Omega|$
  - High frequency: response decay with frequency, $\chi \sim K - 1 \sim \omega^{-2}$ like an electron gas.

Medium of oscillators (6)

- What happens in the limit when the damping Γ goes to zero?
- Again assume $\omega \sim \Omega$ then

$$K \approx 1 - \frac{\omega_p^2}{2\Omega}\frac{1}{(\omega - \Omega + i\Gamma/2)}$$

- The limit where Γ goes to zero can be rewritten using the Plemej formula

$$\lim_{\Gamma\to 0} K \approx \lim_{\Gamma\to 0}\left(1 - \frac{\omega_p^2}{2\Omega}\frac{1}{(\omega - \Omega + i\Gamma/2)}\right) = 1 - \frac{\omega_p^2}{2\Omega}\frac{1}{(\omega - \Omega + i0)} =$$

$$= 1 - \frac{\omega_p^2}{2\Omega}\left[\wp\frac{1}{\omega - \Omega} - i\pi\delta(\omega - \Omega)\right]$$