Homework 3

Finite Differences and Absolute Stability

due February 11, 2016

Task 1: Finite Difference Scheme

Find the highest order finite differences approximation possible of the first derivative of $u(x)$ at the grid nodes $x = x_i$ based on four grid values u_{i-1}, u_i, u_{i+1} and u_{i+2}, where $u_i := u(x_i)$. Assume equidistant grid spacing, *i.e.* $\Delta x := x_{i+1} - x_i = x_i - x_{i-1}$, for all i.

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_i} \approx f(u_{i-1}, u_i, u_{i+1}, u_{i+2})$$

a) Give the approximation of the derivative.

b) What is the leading error term? What is the order of this scheme?

c) Implement this scheme for the approximation of the derivative in a similar way as you did in Task 2a) of Homework 1 and numerically assess the order of its accuracy (Hint. You should consider the slope of the discretization error in the log-log plot). Note that you do not need to compute truncation and round-off errors separately, just the global order of accuracy is required.

Task 2: Stability Criterion

The range of absolute stability of the linear multistep 2^{nd} order Adams–Bashforth method is studied. This time-stepping method for an initial value problem of the form $u'(t) = f(u,t),\ u(t_0) = u_0$ is:

$$u^{n+1} = u^n + \frac{\Delta t}{2}\left[3f^n - f^{n-1}\right],$$

where

$$f^n := f(u^n, t^n), \text{ and } t^n := n\Delta t.$$

a) Is this an explicit or implicit method? Why?

b) Consider, like in Homework 1, the simple test equation (Dahlquist equation):

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= \lambda u = f(u,t), \qquad \lambda \in \mathbb{C},\, t \geq 0, \qquad (1)\\ u(0) &= 1.\end{aligned}$$

Derive for the 2^{nd} order Adams–Bashforth representation of Eq. (1) the amplification factor $G(\lambda\Delta t) = u^{n+1}/u^n$, and plot with MATLAB the stability region $|G(\lambda\Delta t)| \leq 1$ in the complex plane. Plot also the stability region of the Explicit Euler method ($G(\lambda\Delta t) = 1 + \lambda\Delta t$) and compare the two. What can you conclude from the plots? Why and when should you use the 2^{nd} order Adams–Bashforth method instead of the Euler Explicit method?

Task 3: The Modified Wavenumber

On an equispaced grid, the finite-difference derivative of a Fourier mode e^{ikx} can be found by multiplying the function value on each node with the so-called modified wavenumber $\tilde{k}(k)$.

To better understand this concept consider a periodic function

$$f(x + pm) = f(x), \quad m \in \mathbb{Z}.$$

where p is the period length. Let $\underline{f}$ be the discrete representation of $f(x)$ on the equidistant grid where $x_j := j\Delta x$, $\Delta x := p/N$, $j = 0, 1, \ldots, N-1$,

$$\underline{f} := [f_0, f_1, \ldots, f_{N-1}]^\top, \quad f_0 = f_N, \quad \text{where } f_j := f(x_j).$$

For this task consider $p = 2\pi$ and $N = 20$.

a) A first order right-sided finite differences discretization of the derivative $f'(x)$ can be written as

$$\underline{f}'_{num} := [\delta f_0, \delta f_1, \ldots, \delta f_{N-1}]^\top = \underline{\underline{D}}\ \underline{f},$$

where

$$\delta f_j := \frac{f_{j+1} - f_j}{\Delta x}.$$

Use MATLAB to assemble the system matrix $\underline{\underline{D}}$ (remember that $f_0 = f_N$). Include $\underline{\underline{D}}$ in the written report.

b) Consider $f(x) = e^{ikx}$ and derive the expression for the modified wavenumber $\tilde{k}$ for the right-sided finite-difference scheme. Non-dimensionalise the wavenumber with the grid spacing, *i.e.* derive the expression for $\tilde{k}\Delta x$.

c) From now on assume that $k = 5$ (*i.e.* consider a specific wave). Compute the derivative in a discrete (δf_j) and analytical ($f'(x)|_{x=x_j}$) manner at every grid point. Use the previously defined $\underline{\underline{D}}$ for the discrete derivative. Plot the real part for both the numerical and the analytical derivative as a function of x.

d) Compute the vector $\underline{\mu}$ with the elements

$$\mu_j := \frac{\delta f_j}{f_j}$$

and compare it with the complex number $i\tilde{k}$, where $\tilde{k}$ is the modified wavenumber for the right-sided finite differences as derived in b). Does this result confirm that the finite-difference derivative of a Fourier mode e^{ikx} can be found by multiplying the function by the modified wavenumber? *i.e.* does $i\tilde{k}\underline{f} = \underline{\underline{D}}\ \underline{f}$ hold?