Wave equations and properties of waves in ideal media

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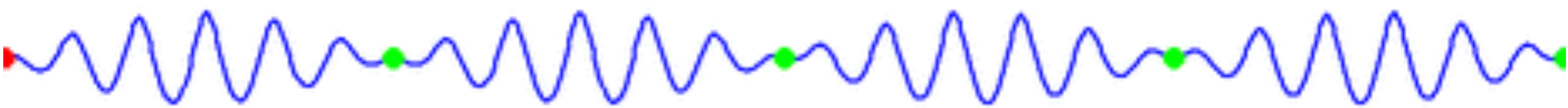
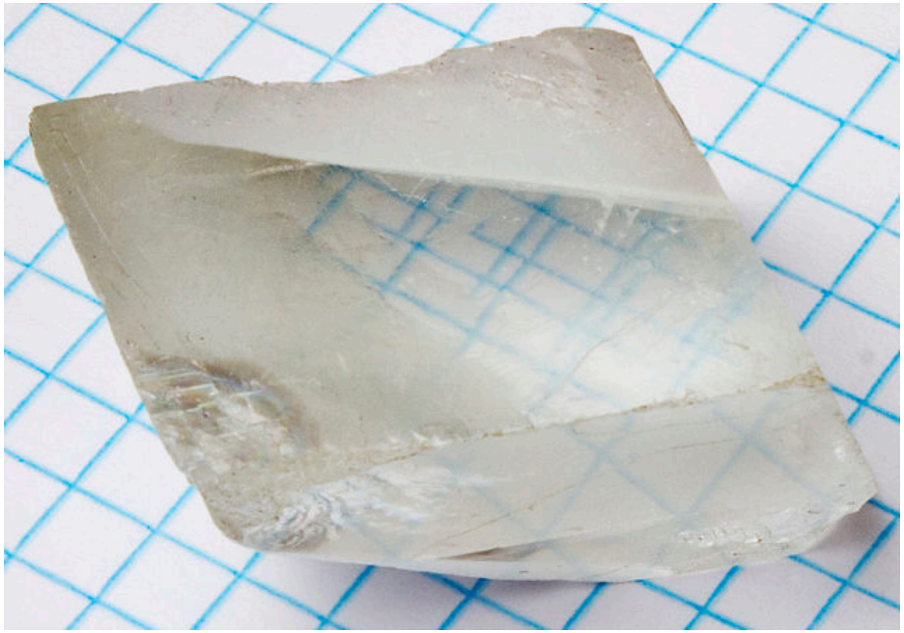


Outline

- The wave equation in vacuum – define:
  - Dispersion equation / dispersion relations / refractive index
- Wave equations in dispersive media
  - Generalise the dispersion equation as a non-linear eigenvalue problem
  - Polarisation vectors and longitudinal/transverse waves
- Damping rates and the antihermitian part of the dielectric tensor
  - Some math for wave equations; mainly linear algebra
- The group velocity
- Waves in ideal media
  - Isotropic media
  - Spatial dispersive media
  - Uniaxial crystals
    E.g. birefringent crystals



The wave equation in vacuum

- Wave equations can be derived for **B**, **E** and **A**.
- **Waves in vacuum**, i.e. no free charge or currents; then ϕ=*const*! Using Fourier transformed quantities in the Coloumb gauge:

$$\mathbf{E}(\omega,\mathbf{k}) = i\omega\mathbf{A}(\omega,\mathbf{k}) \quad , \quad \mathbf{B}(\omega,\mathbf{k}) = i\mathbf{k}\times\mathbf{A}(\omega,\mathbf{k}) \quad , \quad i\mathbf{k}\bullet\mathbf{A}(\omega,\mathbf{k}) = 0$$

- Ampere's law:

$$i\mathbf{k}\times\mathbf{B} + i\omega\mathbf{E}/c^2 = \mu_0\mathbf{J} \quad \Rightarrow \quad \mathbf{k}\times(\mathbf{k}\times\mathbf{A}) + \omega^2/c^2\mathbf{A} = -\mu_0\mathbf{J}$$

where $\mathbf{k}\times(\mathbf{k}\times\mathbf{A}) = \mathbf{k}(\mathbf{k}\bullet\mathbf{A}) - |\mathbf{k}|^2\mathbf{A} = -|\mathbf{k}|^2\mathbf{A}$

- Homogeneous wave equation:

$$\left(|\mathbf{k}|^2 - \omega^2/c^2\right)\mathbf{A} = 0$$

- Solutions exists for: $\left(|\mathbf{k}|^2 - \omega^2/c^2\right) = 0$, the *dispersion equation*!

Dispersion relations

- A wave satisfying a dispersion equation is called a *Wave Mode*.

- Solutions to the dispersion equation can be written as a relation between ω and $\mathbf{k}$ called a *dispersion relation*, e.g.

$$\left(|\mathbf{k}|^2 - \omega^2/c^2\right) = 0 \quad \Rightarrow \quad \omega = \omega_M(\mathbf{k})$$

  - Note: here ω is the frequency and $\omega_M(\mathbf{k})$ is a function of $\mathbf{k}$
  - the sub-index M is for wave mode.
  - the function ω_M represents the dielectric response and therefore is a property of the media

- **Example**: In vacuum the dispersion relation reads:

$$\omega = \pm|\mathbf{k}|c \quad \Rightarrow \quad \omega_{M\pm}(\mathbf{k}) = \pm|\mathbf{k}|c$$

i.e. light waves

Refractive index

- Dispersion relations can be written using the refractive index n

$$n \equiv \frac{|\mathbf{k}|c}{\omega} = \frac{c}{\omega/|\mathbf{k}|} \sim \frac{\text{"speed of light"}}{\text{"phase velocity"}}$$

- A dispersion relation for a wave mode can be rewritten…
  - by replacing $\omega^2 = (|\mathbf{k}|c/n)^2$

$$n \equiv n_M(\mathbf{k})$$

  - or by replacing $\mathbf{k} = k\mathbf{e}_k \;,\; k = \omega n/c$

$$n \equiv n_M(\omega, \mathbf{e}_k)$$

- The dispersion relation for waves in vacuum then reads

$$n = \pm 1$$

i.e. the *phase velocity* of vacuum waves is the *speed of light*

Degrees of freedom of the plane wave

- The solution to the wave equation is a sum of plane waves.

$$A_i(\mathbf{x},t) = \hat{A}_i \exp(i\mathbf{k}\bullet\mathbf{x} - i\omega t)$$

- The dispersion relation determines the refractive index
  - i.e. a relation between ω and k for a given direction of propagation

$$A_i(\mathbf{x},t) = \hat{A}_i \exp(i\mathbf{k}\bullet\mathbf{x} - i\omega_M(\mathbf{k})t)$$

- Is there anything else we can say about the solution?

- **Example**: If the wave is launched by an antenna, which degrees of freedom are determined by
  - the antenna?
  - the media?

The wave equation in dispersive media

- Ex: Temporal Gauge, ϕ=0, the fields are described by **A** alone

$$\mathbf{E}(\omega,\mathbf{k}) = i\omega\mathbf{A}(\omega,\mathbf{k}) \quad , \quad \mathbf{B}(\omega,\mathbf{k}) = i\mathbf{k}\times\mathbf{A}(\omega,\mathbf{k})$$

- Ampere's law:

$$i\mathbf{k}\times\mathbf{B} + i\omega\mathbf{E}/c^2 = \mu_0\mathbf{J} \quad \Rightarrow \quad \mathbf{k}\times(\mathbf{k}\times\mathbf{A}) + (\omega/c)^2\mathbf{A} = -\mu_0\mathbf{J}$$

- Split $\mathbf{J}=\mathbf{J}_{\mathrm{ext}}+\mathbf{J}_{\mathrm{ind}}$, where $\mathbf{J}_{\mathrm{ext}}$ external drive and $\mathbf{J}_{\mathrm{ind}}$ is induced parts

$$J_{\mathrm{ind},\,i} = \alpha_{ij}A_j$$

where α_{ij} is *polarisation response tensor*

- Inhomogeneous wave equation:

Wave operator

$$\Lambda_{ij}A_j = -\frac{c^2}{\omega^2}\mu_0 J_{\mathrm{exp},\,i} \qquad \text{where} \quad \Lambda_{ij} = \frac{c^2}{\omega^2}\left(k_ik_j - |\mathbf{k}|^2\delta_{ij}\right) + K_{ij}$$

$\mathbf{k}\times\mathbf{k}\times\ldots$

Dielectric tensor: $K_{ij} = \delta_{ij} + \dfrac{1}{\omega^2\varepsilon_0}\alpha_{ij}$

Dispersion relations in dispersive media

- Homogeneous wave equation:

$$\Lambda_{ij}(\omega,\mathbf{k})A_j(\omega,\mathbf{k}) = 0$$

(*the book includes only the Hermitian part Λ^H, but this is a technicality At the end of this calculations we get the same dispersion relation*)

- Solutions exist if and only if:

$$\Lambda(\omega,\mathbf{k}) \equiv \det\left[\Lambda_{ij}(\omega,\mathbf{k})\right] = 0$$

this is the *dispersion equation*.

- From this equation the *dispersion relation* can be derived

$$\omega = \omega_M(\mathbf{k})$$

where

$$\Lambda\left(\omega_M(\mathbf{k}),\mathbf{k}\right) = 0$$

Non-linear and linear eigenvalue problems

- This wave equation is a *non-linear eigenvalue problem*…meaning?
- Remember *linear eigenvalue problems*:
  for a matrix $\mathbf{A}$ find the eigenvalues λ and the eigenvectors $\mathbf{x}$ such that:

$$\mathbf{A}\mathbf{x} - \lambda\mathbf{x} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

  or alternatively

$$(A_{ij} - \lambda\delta_{ij})x_j = \Lambda_{ij}(\lambda)x_j = 0$$

  Thus for the linear eigenvalue problem Λ_{ij} is linear in λ .
- Our wave equation has the same form, except $\Lambda_{ij}(\omega)$ is non-linear in ω .
- Thus, we are looking for the eigenvalues ω_M and the eigenvectors $\mathbf{A}$ to the equation

$$\Lambda_{ij}(\omega_M, \mathbf{k})A_j = 0$$

- **Exercise**: show that when $K_{ij}=K_{ij}(\mathbf{k})$, the wave equation is a linear eigenvalue problem in ω^2. However, inertia in Eq. of motion (when deriving media responce) gives $K_{ij} = K_{ij}(\omega,\mathbf{k})$.

Polarization vector

- So the wave equation is an eigenvalue problem
  - The eigenvalue is the frequency
  - The normalised eigenvector is called the *polarisation vector,* $\mathbf{e}_M(\mathbf{k})$

$$\mathbf{e}_M(\mathbf{k}) = \frac{\mathbf{A}\left(\omega_M(\mathbf{k}),\mathbf{k}\right)}{\left|\mathbf{A}\left(\omega_M(\mathbf{k}),\mathbf{k}\right)\right|}$$

the direction of the $\mathbf{A}$-field!

  - *Note*: the $\mathbf{A}$-field is parallel to the $\mathbf{E}$-field
- The polarisation vector is complex – what does this mean?
  - e.g. take $\mathbf{e}_M = (\,2\,,\,\mathrm{i}\,,0\,)\,/\,5^{1/2}$, then the vector potential is

$$\mathbf{A}(t,\mathbf{x}) \propto \mathrm{Re}\left\{\,[\;2\;,\;i\;,\;0\;]\;\exp(i\mathbf{k}\cdot\mathbf{x}+i\omega t)\right\} =$$

$$= \left[\;2\cos(\mathbf{k}\cdot\mathbf{x}+\omega t)\;,\;\cos(\mathbf{k}\cdot\mathbf{x}+\omega t+90^{o})\;,\;0\;\right]$$

  - The difference in "phase" of e_{M1} and e_{M2} (in complex plane; one being real and the other imaginary) makes A_1 and A_2 oscillate *90ᵒ* out of phase – elliptic polarisation!

Longitudinal & Transverse waves

Definition:

Longitudinal & Transverse waves have $\mathbf{e}_M$ parallel & perpendicular to $\mathbf{k}$

Examples:

- *Light waves* have $\mathbf{E} \parallel \mathbf{A}$ perpendicular to $\mathbf{k}$, i.e. a ***transverse*** wave
- *Sounds waves* (wave equation for the fluid velocity $\mathbf{v}$) have $\mathbf{v} \parallel \mathbf{k}$, i.e. a ***longitudinal*** wave.

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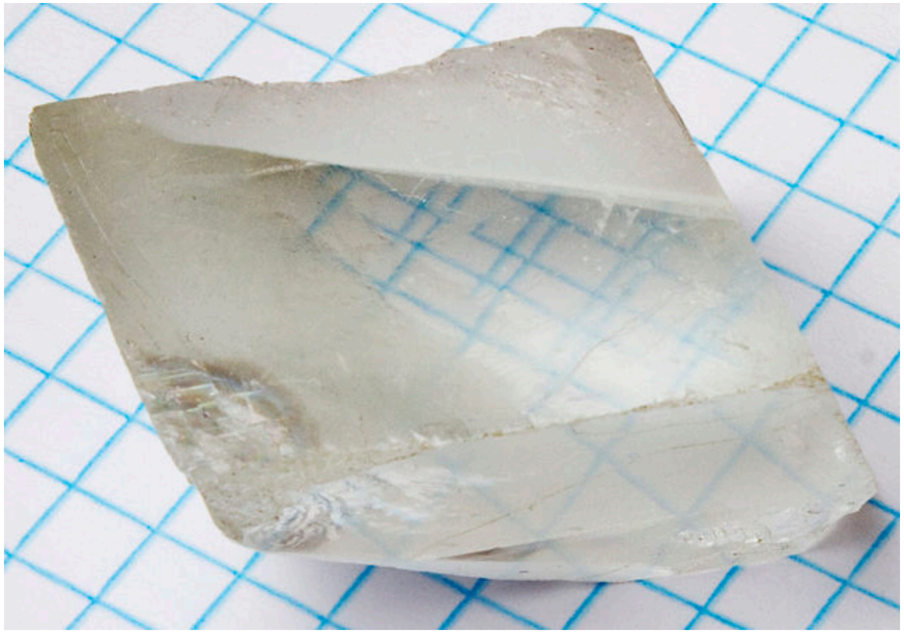

Linear algebra: cofactors

- An (i,j):th *cofactor*, λ_{ij} of a matrix $\mathbf{\Lambda}$ is the determinant of the "reduced" matrix, obtained by removing row i and column j , times $(-1)^{i+j}$
- In tensor notation (*you don't have to understand why!*):

$$\lambda_{ai} = \frac{1}{2}\varepsilon_{abc}\varepsilon_{ijl}\Lambda_{bj}\Lambda_{cl} \quad \text{e.g.} \quad \lambda_{21} = (-1)^{i+j}\det\begin{vmatrix} * & \Lambda_{12} & \Lambda_{13} \\ * & * & * \\ * & \Lambda_{32} & \Lambda_{33} \end{vmatrix} = (-1)^{i+j}\begin{vmatrix} \Lambda_{12} & \Lambda_{13} \\ \Lambda_{32} & \Lambda_{33} \end{vmatrix}$$

reduced matrix

- Alternative definition for cofactors:

$$\Lambda_{ik}\lambda_{kj} = \Lambda\delta_{ij}$$

  – Thus, for $\Lambda=0$ each column $(\lambda_{1j}, \lambda_{2j}, \lambda_{3j})^T$ is an eigenvector!
- It can be shown that

$$\lambda_{ai} = \lambda_{kk}e_{Mi}e^{*}_{Mj}$$

where λ_{kk} is the *trace* of λ and e_{Mi} are the normalised eigenvectors

Linear algebra: determinants ★

- The determinant can be written as (Melrose page 139)

$$\det[\Lambda] = \frac{1}{6}\varepsilon_{abc}\varepsilon_{ijl}\Lambda_{ai}\Lambda_{bj}\Lambda_{cl}$$

- Derivatives (note that the three derivates are identical)

$$\frac{\partial}{\partial x}\det[\Lambda(x)] = \underbrace{\frac{1}{2}\varepsilon_{abc}\varepsilon_{ijl}\Lambda_{ai}\Lambda_{bj}}_{\text{Cofactors } \lambda_{bj}!}\frac{\partial \Lambda_{cl}}{\partial x} = \lambda_{bj}\frac{\partial \Lambda_{bj}}{\partial x}$$

- Special case; take derivative w.r.t. the one tensor component

$$\frac{\partial}{\partial \Lambda_{ij}}\det[\Lambda(\Lambda_{11},\Lambda_{12},\Lambda_{21},\Lambda_{22}\ldots)] = \lambda_{nm}\underbrace{\frac{\partial \Lambda_{nm}}{\partial \Lambda_{ij}}}_{\delta_{ni}\delta_{jm}} = \lambda_{ij}$$

Linear algebra: Taylor expansion ★

- The determinant of this matrix is a function of the matrix components

$$\det[\Lambda] = f(\Lambda_{11}, \Lambda_{12}, \ldots)$$

- Perturbing the matrix components $\Lambda_{ij} \to \Lambda_{ij} + \delta\Lambda_{ij}$ we can then Taylor expand

$$\det[\Lambda + \delta\Lambda] = f(\Lambda_{ij} + \delta\Lambda_{ij}) =$$

$$= f(\Lambda_{ij}) + \frac{\partial}{\partial \Lambda_{ij}} f(\Lambda_{ij})\delta\Lambda_{ij} + O(\delta\Lambda^2) =$$

$$= \det[\Lambda] + \frac{\partial}{\partial \Lambda_{ij}} \det[\Lambda]\delta\Lambda_{ij} + O(\delta\Lambda^2) =$$

$$= \det[\Lambda] + \lambda_{ij}\delta\Lambda_{ij} + O(\delta\Lambda^2)$$

Damping of waves

- We're now ready to show that for low amplitude waves:
  - the Hermitian part of Λ_{ij} provides the dispersion relation
  - the anti-Hermitian part of the dielectric tensor $K^A{}_{ij}$ describes wave damping, i.e. the decay of the wave
- Consider a plane wave with complex frequency $\omega+i\omega_I$

$$A_i(\mathbf{x},t) = \hat{A}_i \exp(\omega_I t)\exp(i\mathbf{k}\bullet\mathbf{x} - i\omega t)$$

  - The wave amplitude decays at a rate $-\omega_I$
  - Note: the wave energy ($\sim|\mathbf{E}|^2$) decays at a rate $\gamma=-2\omega_I$
- The dispersion relation

$$\det\left[\Lambda_{ij}(\omega + i\omega_I,\mathbf{k})\right] = \det\left[\Lambda^H_{ij}(\omega + i\omega_I,\mathbf{k}) + \Lambda^A_{ij}(\omega + i\omega_I,\mathbf{k})\right] = 0$$

Exercise: show that $\mathbf{\Lambda}^A=\mathbf{K}^A$

- To simplify this expression we will assume *weak damping* …

Weak damping of waves

- Assume the damping to be weak by:

$$K_{ij}^{A} \rightarrow 0 \quad \text{and} \quad \omega_I \rightarrow 0$$

- Also assume $\omega_I \sim K^A$
  - Interpretation of the relation $\omega_I \sim K^A$: reduce K^A by factor, then ω_I reduces by the same factor, thus the they go to zero *together*
- Expand in small ω_I :

$$\Lambda_{ij}(\omega + i\omega_I, \mathbf{k}) \approx \Lambda_{ij}(\omega, \mathbf{k}) + i\omega_I \frac{\partial}{\partial \omega} \Lambda_{ij}(\omega, \mathbf{k}) + O(\omega_I^{\,2})$$

$$\approx \Lambda_{ij}^{H}(\omega, \mathbf{k}) + K_{ij}^{A}(\omega, \mathbf{k}) + i\omega_I \frac{\partial}{\partial \omega} \Lambda_{ij}^{H}(\omega, \mathbf{k}) + i\omega_I \frac{\partial}{\partial \omega} K_{ij}^{A}(\omega, \mathbf{k}) + O(\omega_I^{\,2})$$

1st order in ω_I

Both small, i.e. $\sim \omega_I^2$

- Dispersion equation then reads

$$\det\left[\Lambda_{ij}^{H} + \delta\Lambda_{ij}\right] = 0 \quad , \quad \delta\Lambda_{ij} = K_{ij}^{A}(\omega, \mathbf{k}) + i\omega_I \frac{\partial \Lambda_{ij}^{H}(\omega, \mathbf{k})}{\partial \omega} + O(\omega_I^{\,2})$$

  - Next: Expand the determinant in small $\delta\Lambda_{ij}$

Weak damping of waves

- The dispersion equation (repeated from previous page):

$$\det\left[\Lambda_{ij}^{H} + \delta\Lambda_{ij}\right] = 0 \quad , \quad \delta\Lambda_{ij} = K_{ij}^{A}(\omega,\mathbf{k}) + i\omega_I \frac{\partial \Lambda_{ij}^{H}(\omega,\mathbf{k})}{\partial \omega} + O(\omega_I{}^2)$$

- Taylor expand the determinant

$$\det\left[\Lambda_{ij}^{H} + \delta\Lambda_{ij}\right] = \det\left[\Lambda_{ij}^{H}\right] + \delta\Lambda_{ij}\lambda_{ij} + O\left(\delta\Lambda_{ij}{}^2\right)$$

  – where λ_{ij} are the cofactors of

- NOTE: (see "Linear Algebra" pages): $\lambda_{ij} \dfrac{\partial}{\partial \omega} \Lambda_{ij}^{H}(\omega,\mathbf{k}) = \dfrac{\partial}{\partial \omega} \det\left[\Lambda_{ij}^{H}\right]$

- The dispersion equation can then be written as

$$\det\left[\Lambda_{ij}^{H}(\omega,\mathbf{k})\right] + \lambda_{ij} K_{ij}^{A}(\omega,\mathbf{k}) + i\omega_I \frac{\partial}{\partial \omega} \det\left[\Lambda_{ij}^{H}(\omega,\mathbf{k})\right] + O(\omega_I{}^2) = 0$$

Weak damping of waves

- Note that the dispersion equation with weak damping has both *real* and *imaginary* parts
  - The matrix of cofactors is Hermitian, thus $\lambda_{ij} K_{ij}^{A}$ is imaginary
  - Also: $\det(\Lambda_{ij}{}^{H})$ is real

$$0 = \mathrm{Re}\left\{\det\left[\Lambda(\omega,\mathbf{k})\right]\right\} \approx \det\left[\Lambda_{ij}^{H}(\omega,\mathbf{k})\right] + O(\omega_I{}^2)$$

$$0 = \mathrm{Im}\left\{\det\left[\Lambda(\omega,\mathbf{k})\right]\right\} \approx -i\lambda_{ij}K_{ij}^{A}(\omega,\mathbf{k}) + \omega_I \frac{\partial}{\partial\omega}\det\left[\Lambda_{ij}^{H}(\omega,\mathbf{k})\right] + O(\omega_I{}^2)$$

- The first equation gives dispersion relation for real frequency

$$\omega = \omega_M(\mathbf{k}) \quad \text{such that :} \quad \det\left[\Lambda_{ij}^{H}(\omega_M(\mathbf{k}),\mathbf{k})\right] + O(\omega_I{}^2) = 0$$

and the second equations gives the damping rate

$$\omega_I = \frac{i\lambda_{ij}K_{ij}^{A}(\omega_M(\mathbf{k}),\mathbf{k})}{\frac{\partial}{\partial\omega}\det\left[\Lambda_{nm}^{H}(\omega,\mathbf{k})\right]_{\omega=\omega_M(\mathbf{k})}} + O(\omega_I{}^2)$$

Energy dissipation rate, γ_M

- Alternatively we can form the *energy dissipation rate*, i.e. rate at which the wave energy is damped $\gamma_M = -2\omega_I$
  - express the cofactor in terms of polarisation vectors $\lambda_{ij} = \lambda_{kk} e_{Mi} e^*_{Mj}$

$$\gamma_M = -2i\omega_M(\mathbf{k}) R_M(\mathbf{k}) \{ \underbrace{e^*_{Mi}}_{\text{Vector}}(\mathbf{k}) \underbrace{K^A_{ij}}_{\text{Matrix}}(\omega_M(\mathbf{k}),\mathbf{k}) \underbrace{e_{Mj}}_{\text{Vector}}(\mathbf{k}) \}$$

Note: this is related to the hermitian part of the conductivity, $\sigma^H_{ij} \propto iK^A_{ij}$

$$\gamma_M \propto e^*_{Mi} \left[iK^A_{ij} \right] e_{Mj} \propto e^*_{Mi} \sigma^H_{ij} e_{Mj}$$

  - here R_M is the *ratio of electric to total energy*

$$R_M(\mathbf{k}) = \left\{ \frac{\lambda_{ss}(\omega,\mathbf{k})}{\omega \frac{\partial}{\partial\omega} \det\left[\Lambda^H_{nm}(\omega,\mathbf{k})\right]} \right\}_{\omega=\omega_M(\mathbf{k})}$$

and plays an important role in Chapter 15

Determinant and the cofactors in the general case ★

- Explicit forms of dispersion equation and cofactors
- Write $\mathbf{\Lambda}$ in terms of the refractive index n and the unit vector along $\mathbf{k}$, i.e. $\boldsymbol{\kappa} = \mathbf{k}/|\mathbf{k}|$

$$\Lambda_{ij} = \frac{c^2}{\omega^2}\left(k_i k_j - |\mathbf{k}|^2 \delta_{ij}\right) + K_{ij} \quad \rightarrow \quad \Lambda_{ij} = n^2\left(\kappa_i \kappa_j - \delta_{ij}\right) + K_{ij}$$

- Brute force evaluation give

$$\det[\Lambda] = n^4 \kappa_i \kappa_j K_{ij} - n^2\left(\kappa_i \kappa_j K_{ij} K_{ss} - \kappa_i \kappa_j K_{is} K_{sj}\right) + \det[K]$$

- The cofactors (related to the eigenvector) are

$$\lambda_{ij} \approx n^4 \kappa_i \kappa_j - n^2\left(\kappa_i \kappa_j K_{ss} - \delta_{ij} \kappa_r \kappa_s K_{rs} - \kappa_i \kappa_s K_{sj} - \kappa_s \kappa_j K_{is}\right) +$$
$$+ \tfrac{1}{2} \delta_{ij}\left(K_{ss}{}^2 - K_{rs} K_{sr}\right) + K_{is} K_{sj} + K_{ss} K_{ij}$$

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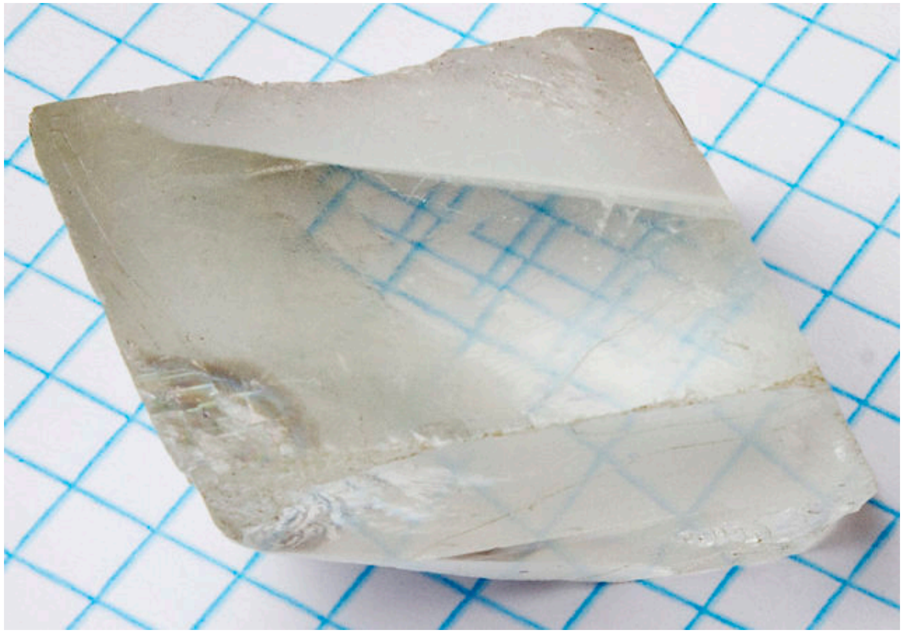
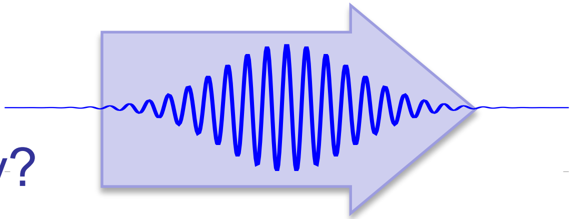


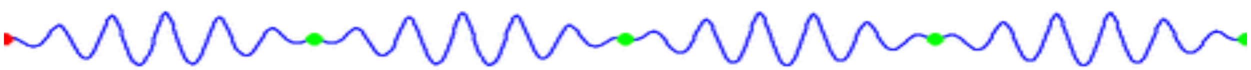
The group velocity

- The propagation of waves is a transfer of energy
  - e.g. the light from the sun transfer energy to earth
    - It’s warm in the sun since the sunlight bring energy!

- Consider a wave package from an antenna
  - Does this package travel with the phase velocity?



In dispersive media the answer is **no**!!



"Wave group" by Kraaiennest - Own work. Licensed under GFDL via Commons - https://commons.wikimedia.org/wiki/File:Wave_group.gif#/media/File:Wave_gr oup.gif

The velocity at which the shape of the wave's amplitudes (modulation/envelope) moves is called the **group velocity**

- The group velocity is *often* the velocity of information or energy
  - *Warning*! There are exceptions; experiments have shown that group velocity can go above speed of light, but then the information does not travel as fast

The velocity of a wave package, 1(2)

- The concept of group velocity can be illustrated by the motion of a wave package
  - This motion can easily be identified for a 1D wave package
  - travelling in a wave mode with dispersion relation: $\omega = \omega_M(k)$
  - assuming the wave is *almost monochromatic* – linearise the dispersion relation:

$$\omega_M(k) \approx \omega_{M0} + \omega'_{M0}(k - k_0) \quad , \quad \omega'_{M0} \equiv \frac{d\omega_{M0}}{dk}$$

- Let the wave have a Fourier transform

complex conjugate of the first term: below denoted c.c.

$$E(\omega,k) = A(k)\delta\big(\omega - \omega_M(k)\big) + \underbrace{A^*(-k)\delta\big(\omega + \omega_M(k)\big)}$$

- To study how the wave package travel in space-time, take the inverse Fourier transform

$$E(t,r) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk A(k)\delta\big(\omega - \omega_M(k)\big)\exp\{ikx - i\omega t\} + \text{c.c.}$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} dk A(k)\exp\{ikx - i\omega_M(k)t\} + \text{c.c.}$$

The velocity of a wave package, 2(2)

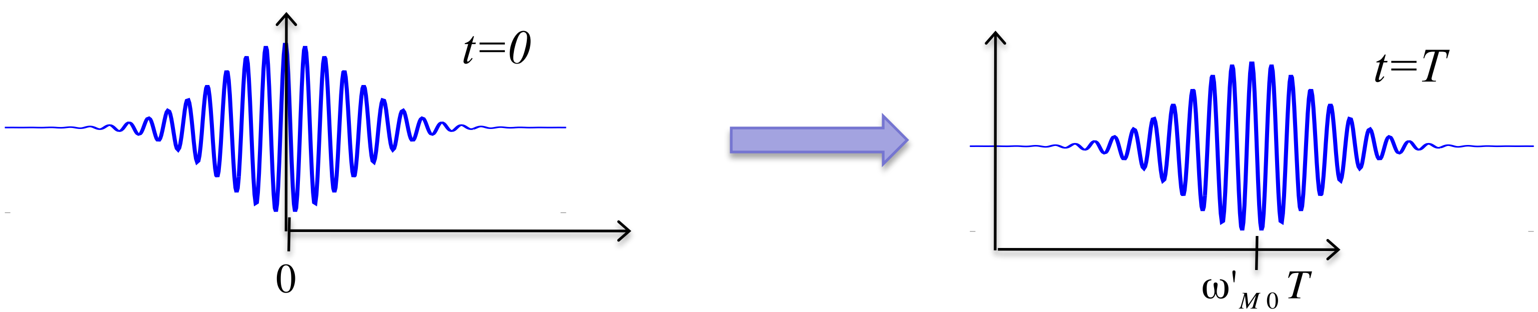
- Now apply the assumption of having "almost chromatic waves"

$$\omega_M(k) \approx \omega_{M0} + \omega'_{M0}(k - k_0) \Rightarrow$$

$$E(t,x) \approx \frac{1}{4\pi}\int_{-\infty}^{\infty} dk A(k) \exp\{ikx - i\omega_M(k)t\} + \text{c.c.} =$$

$$= \frac{e^{-i(\omega_{M0}+\omega'_{M0}k_0)t}}{4\pi}\int_{-\infty}^{\infty} dk A(k) \exp\{ik(x - \omega'_{M0}t)\} + \text{c.c.} = e^{-i(\omega_{M0}+\omega'_{M0}k_0)t} fcn(x - \omega'_{M0}t) + \text{c.c.}$$

  – i.e. if a wave package is centered around $x=0$ at time $t=0$, then at time $t=T$ wave package has the identical shape but now centred around $x = \omega'_{M0}T$



- Wave package moves with a speed called the *group velocity* :

$$v_g = \omega'_{M0} \equiv \frac{d\omega_{M0}}{dk}$$

Examples of group velocities

- Let us start with the ordinary light wave:

$$\omega_L(\mathbf{k}) = ck = c\sqrt{k_i k_i}$$

  - The group velocity:

$$v_{gM,i} \equiv \frac{\partial}{\partial k_i}\omega_L(\mathbf{k}) = \frac{\partial}{\partial k_i}ck = c\kappa_i$$

  - The phase velocity:

$$v_{phM,i} \equiv \frac{\omega_L(\mathbf{k})}{k}\kappa_i = c\kappa_i$$

Hamilton's equations of motion for rays ★

- The concept of group velocity can be used when studying rays
- How do you follow the path of a ray in a dispersive media?
  - Hamilton studied this problem in the mid 1800's and developed a ***particle theory for waves***; i.e. like photons! (long before Einstein)
  - Hamilton's theory is now known as *Hamiltonian mechanics*
  - Hamilton's equations for the mechanical motion of particles:

$$\dot{q}_i(t) = \frac{\partial H(p,q,t)}{\partial p_i}$$

$$\dot{p}_i(t) = -\frac{\partial H(p,q,t)}{\partial q_i}$$

  - where
    - $q_i = (x,y,z)$ are the position coordinates
    - $p_i = (mv_x, mv_y, mv_z)$ are the canonical momentum coordinates
    - The Hamiltonian H is the sum of the kinetic and potential energy
- But what are q_i, p_i and H for waves?

Hamilton's equations for rays

- What are q_i, p_i and H for waves?
  - The position coordinates $q_i = (x,y,z)$
  - In quantum mechanics the wave momentum is $\hbar k$; in Hamilton's theory the momentum is $p_i = (k_x, k_y, k_z)$
  - The Hamiltonian energy H is $\omega_M(k)$ (energy of wave in quanta $\hbar\omega$), i.e. the solution to the dispersion relation for the mode *M*!
- Consequently, the group velocity of a wave-particle of mode *M* is:

$$\mathbf{v}_{gM} \equiv \dot{\mathbf{q}} = \frac{\partial \omega_M(\mathbf{k})}{\partial \mathbf{k}}$$

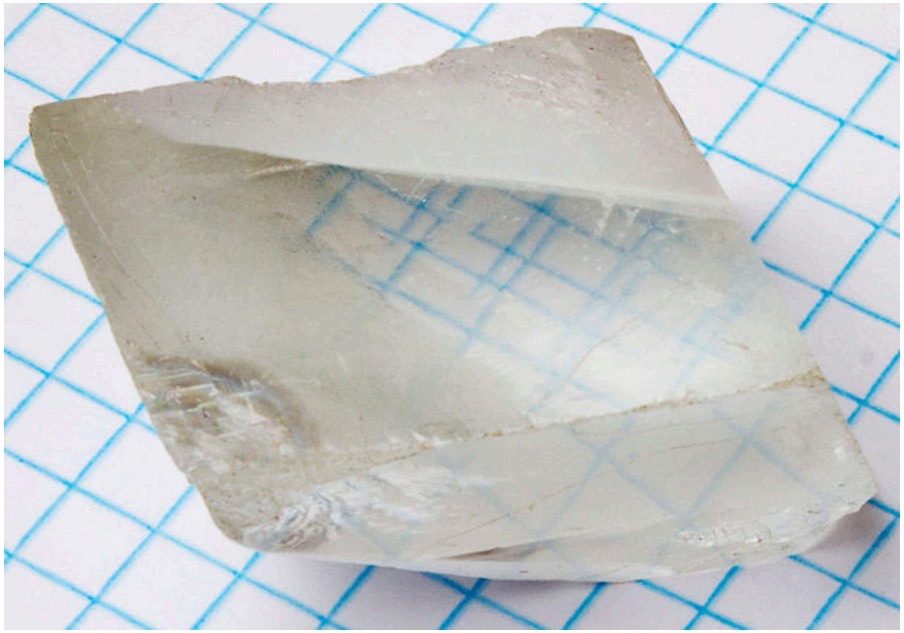
- The second of Hamilton equations tells us how $\mathbf{k}$ changes when passing through a weakly inhomogeneous media, i.e. one in which the dispersion relation changes *slowly* as the wave propagates through the media, $\omega_M(\mathbf{k},\mathbf{q})$

$$\dot{\mathbf{k}} = -\frac{\partial \omega_M(\mathbf{k},\mathbf{q})}{\partial \mathbf{q}}$$

Warning! Hamiltons equations only work for *almost homogeneous media*. If the media changes rapidly the ray description may not work!

Outline

- The wave equation in vacuum – define:
  - Dispersion equation / dispersion relations / refractive index
- Wave equations in dispersive media
  - Generalise the dispersion equation as a non-linear eigenvalue problem
  - Polarisation vectors and longitudinal/transverse waves
- Damping rates and the antihermitian part of the dielectric tensor
  - Some math for wave equations; mainly linear algebra
- The group velocity
- Waves in ideal media
  - Isotropic media
  - Spatial dispersive media
  - Uniaxial crystals
    E.g. birefringent crystals



Ex. 1: Isotropic, not spatially dispersive, media

- Isotropic, not spatially dispersive, media: $K_{ij}(\omega) = K(\omega)\delta_{ij}$

unit vector, $\kappa=\mathbf{k}/k$

- Place z-axis along $\mathbf{k}$: $\Lambda_{ij} = n^2\left(\kappa_i\kappa_j - \delta_{ij}\right) + K_{ij} = \begin{pmatrix} K-n^2 & 0 & 0 \\ 0 & K-n^2 & 0 \\ 0 & 0 & K \end{pmatrix}$
  (n = refractive index)

- Dispersion equation: $\left(K-n^2\right)^2 K = 0$

K is the square root of the **refractive index**

- Dispersion relations: $\begin{cases} n^2 = K(\omega) \rightarrow n_M(\omega)^2 \equiv K(\omega) \\ K(\omega) = 0 \end{cases}$
  - Note: $K(\omega)=0$ means oscillations, NOT waves! (See section on Group velocity)
- The waves $n^2=K(\omega)$ are transverse waves
  - Plug dispersion relation into Λ_{ij} to see that the eigenvectors are perpendicular to $\mathbf{k}$!
- Polarisation vectors of transverse waves are *degenerate* (not unique eigenvector per mode); discussed in detail in Chapter 14.

Ex. 1: Isotropic media – electron gases

- Consider high frequency waves; most medias behave like an electron gas

$$K_{ij} = K\delta_{ij} = \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)\delta_{ij}$$

- High frequency waves:

$$n_M^2 = K(\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

$$\omega_M^2(\mathbf{k}) = c^2k^2 + \omega_{pe}^2$$

  – The group velocity: $v_{g,M,i} = \frac{\partial}{\partial k_i}\sqrt{\omega_{pe}^2 + c^2k^2} = \frac{c}{\sqrt{\omega_{pe}^2/c^2k^2+1}}\kappa_i$

  – The phase velocity: $v_{g,M,i} = \sqrt{\omega_{pe}^2/c^2k^2 + 1}\, c\kappa_i$

  – *Note*:
    - phase velocity may be *faster* than speed of light
    - group velocity is *slower* than speed of light

Information travel with v_g , cannot travel faster than speed of light

Ex 2: Isotropic media with spatial dispersion

- Isotropic media with spatial dispersion; align z-axis: $\mathbf{e}_z = \boldsymbol{\kappa}$

$$K_{ij}(\omega,\mathbf{k}) = K^L(\omega,k)\kappa_i\kappa_j + K^T(\omega,k)\left(\delta_{ij} - \kappa_i\kappa_j\right) = \begin{bmatrix} K^T & 0 & 0 \\ 0 & K^T & 0 \\ 0 & 0 & K^L \end{bmatrix}$$

- Dispersion equation

$$K^L(\omega,k)\left[K^T(\omega,k) - n^2\right]^2 = 0$$

- Transverse dispersion relation $K^T(\omega,k) - n^2 = 0$
  - Again the transverse waves are degenerate.

- The longitudinal dispersion relation $K^L(\omega,k) = 0$
  - Dispersion give us a longitudinal wave!
    (eigenvector parallel to **k**)

Ex 3: Birefringent media

- Uniaxial and biaxial crystals are "*birefringent*"
  - A light ray entering the crystal splits into two rays; the two rays follow different paths through the crystal.
  - Why?
- Consider a uniaxial crystal;
  - align z-axis with the distinctive axis of the crystal

$$K(\omega) = \begin{pmatrix} K_\perp(\omega) & 0 & 0 \\ 0 & K_\perp(\omega) & 0 \\ 0 & 0 & K_\parallel(\omega) \end{pmatrix}$$

  - Align coordinates $\mathbf{k}$ in x-z plane; let θ be the angle between z-axis and $\mathbf{k}$.

$$\kappa = (\sin\theta \ , 0 \ , \ \cos\theta)$$

Birefringent media (cont.)

- Dispersion equation in uniaxial media

$$\left(K_\perp - n^2\right)\left[K_\perp K_\parallel - n^2\left(K_\perp \sin^2\theta + K_\parallel \cos^2\theta\right)\right]^2 = 0$$

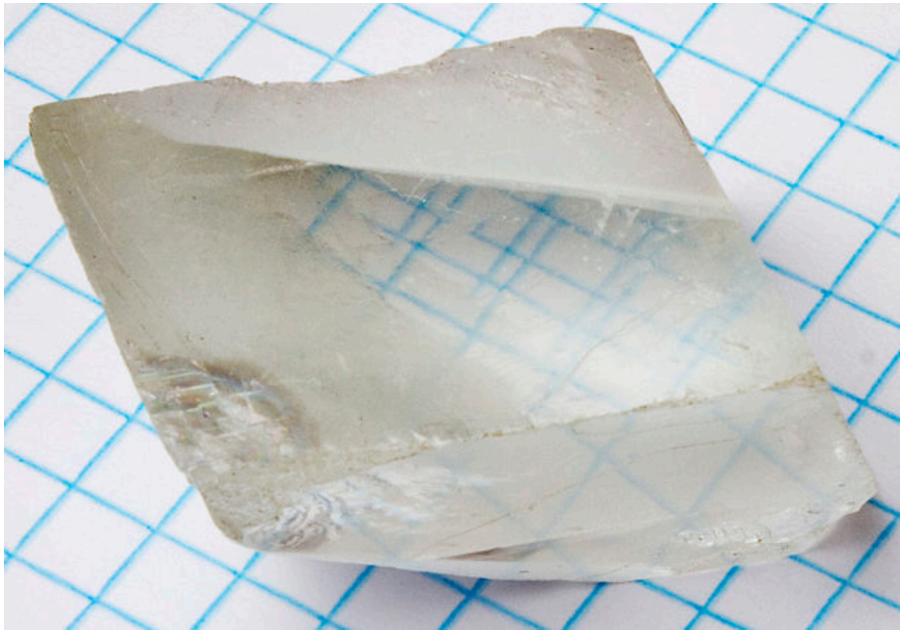
- Two modes, different refractive index (*naming conventions may differ!*)
  - The (ordinary) O-mode: $n_O{}^2 = K_\perp$
  - The (extraordinary) X-mode: $n_X{}^2 = \dfrac{K_\perp K_\parallel}{K_\perp \sin^2\theta + K_\parallel \cos^2\theta}$
- **O-mode**: is transverse: $\mathbf{e}_O(\mathbf{k}) = \left(0\ ,1\ ,0\right)$
  - E-field along the crystal plane
- **X-mode**: is ***not*** transverse and ***not*** longitudinal:

$$\mathbf{e}_X(\mathbf{k}) \propto \left(K_\parallel \cos\theta\ ,\ 0\ ,\ K_\perp \sin\theta\right)$$

  - E-field has components both along and perpendicular to crystal plane

Wave splitting

- Let a light ray fall on a birefringent crystal with electric field components in all directions (x,y,z).
  - The y-component will enter the crystal as an O-mode! (polarisation vector is in y-direction)
  - The x,z-components as X-modes (polarisation vector is in xz-plane)
- The O-mode and X-mode have different refractive index
  - they travel with different speed
  - i.e. the wave will refract differently!



Summary

- We've studied plane wave solutions in dispersive media
  - Plane waves are describes by: $\{\mathbf{E}, \mathbf{k}, \omega\}$
- Wave equation is a eigenvalue problem
  - Dispersion equation
  - Dispersion relation – eigenvalue, $\omega = \omega_M(\mathbf{k})$
  - Polarisation vector – eigenvector, $\boldsymbol{e}_M = \mathbf{E}_M/|\mathbf{E}_M|$
- Common polarisations:
  - Longitudinal $(\mathbf{E} \parallel \mathbf{k})$ & transverse $(\mathbf{E} \perp \mathbf{k})$
  - Elliptical polarisation: complex polaristion vector, e.g. $\boldsymbol{e}_M = [2, i, 0]$
    - Phase difference between x- and y-components
- The group velocity is the velocity of the wave envelope
  - In practice it's also the velocity for transport of energy/information

Summary

- Uniaxial crystals have been shown to be birefringent
  - Two modes, O-mode and X-mode with different refractive indexes, n
  - n depends on angle between $\mathbf{E}$ and normal to crystal plane
  - Incoming waves split into two modes - refract with different angles!