Homework Set 2

Exercise 1 *Let $\bar{X}(T)$ be a forward Euler approximation of the solution to a stochastic differential equation*

$$\begin{aligned} dX(t) &= a(t, X(t))dt + b(t, X(t))dW(t), \\ X(0) &= x_0. \end{aligned}$$

Write a computer program that computes the Forward Euler approximation $\bar{X}$. Test numerically how the strong error

$$\|X(T) - \bar{X}(T)\|_{L^2(\Omega)} = \sqrt{E[(X(T) - \bar{X}(T))^2]}$$

and the weak error

$$E[g(X(T))] - E[g(\bar{X}(T))]$$

depend on the time step Δt, i.e. what the convergence rate is. Try with functions a, b, and g that satisfy the conditions in Theorems 3.1 and 5.8. Also try a function g that does not satisfy the conditions in Theorem 5.8. Can you still observe the same convergence rate?

Exercise 2

a *Consider the ordinary differential equation*

$$dX_t = A\, X_t\, dt$$

where $X_t \in \mathbb{R}^2$ and the matrix A has two real eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -10^5$. Then the backward Euler method

$$X(t_{n+1}) - X(t_n) = AX(t_{n+1})(t_{n+1} - t_n)$$

is an efficient method to solve the problem. Why?

b *Formulate and motivate a backward Euler method for approximation of the Itô SDE*

$$dX_t = aX_t dt + bX_t dW_t;$$

where $a < 0$ and $b > 0$ are constants.