Waves in plasmas

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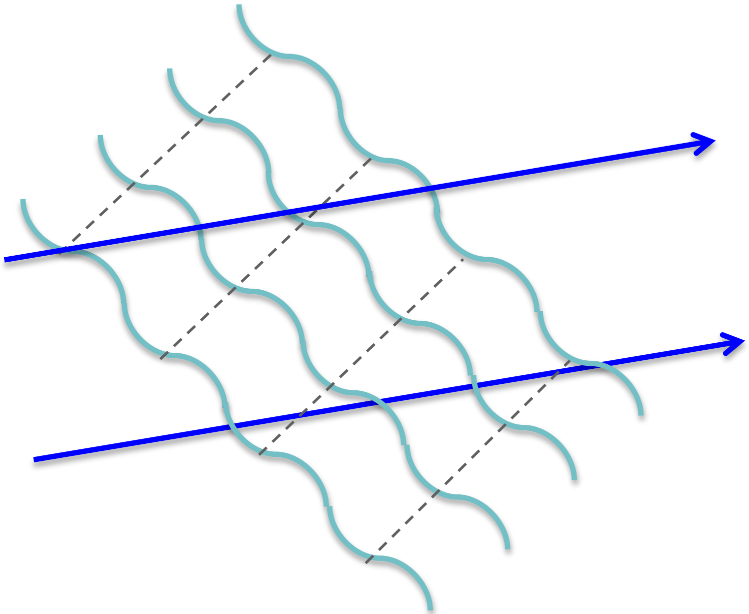
Outline

- Introduction to plasma physics
  - Magneto-Hydro Dynamics, MHD
- Plasmas without magnetic fields
  - Cold plasmas
    - Transverse waves – plasma modified light waves
    - Longitudinal plasma oscillations
  - Warm plasmas – longitudinal waves
    - Debye shielding – incomplete screening at finite temperature
    - Langmuir waves
    - Ion-acoustic (ion-sound) waves
- Alfvén waves - low frequency waves in magnetised plasmas
  - Shear and fast Alfvén waves
- Magnetoionic waves
  - Wave resonances & cut-offs
  - CMA diagram



Introduction of plasma

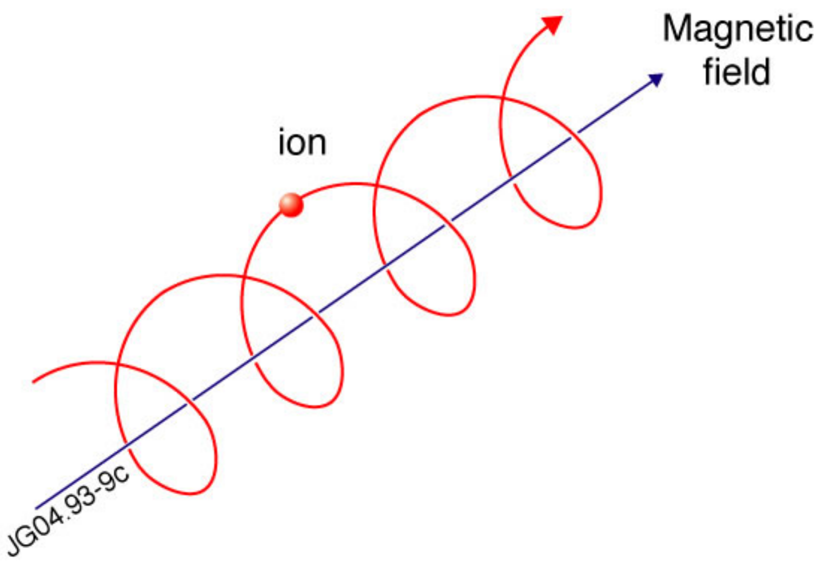
- Plasmas are ionised gases
- High temperature and low concentration; use Newtonian mechanics

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

- High conductivity
  - difficult to produce charge separation – plasmas are quasineutral;
  - except at high frequency, $\omega > \omega_{pe}$, electrons too heavy to react
- Magnetic fields: cause particles to follow gyro orbits
  - $\mathbf{v}\times\mathbf{B}$ force cause particles to gyrate around the magnetic field lines.
  - Gyro frequency: $\Omega = qB/m$
  - Gyro radius: $\rho = v/\Omega = mv/qB$
  - Plasmas can be very anisotropic, in fusion plasmas:

$$\frac{\sigma_{\parallel}}{\sigma_{\perp}} \sim 10^9$$

i.e. it is almost impossible to conduct currents perpendicular to B!



Plasma models

- There are many mathematical plasma models
- We have the *cold plasma* models, i.e. without thermal motion
  - Magneto-ionic theory (inc. electron response, while ions are static)
  - Cold plasma – inc. both electron and ion responses
- And the general *warm plasma* model from lecture 5.
  - This one can also be generalised to include magnetised plasmas.
- Many of these models are too complicated to analyse and solve
  - Almost all practical solutions involve further approximations
  - Typically: expand for a specific range of frequencies, wave lengths, or phase velocity
  - Example:

    $\omega \ll \omega_{pi}$, $\omega \ll \Omega_i$, $V_{th}k \ll \Omega_i \ldots$

    These are the main conditions for the so called *magneto-hydro dynamics* model

The MHD model for a plasma

- Magneto-Hydro Dynamics (MHD), the most famous plasma model!
- It assumes low frequencies compared to the ion cyclotron frequency, $\omega \ll \Omega_i$ and the plasma frequency, $\omega \ll \omega_{pe}$.
- **Assumption**: electrons have an infinite small mass:

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{J}$$

  This equation is referred to as *Ohms law* and η is the resistivity.
  - In the rest frame, $\mathbf{v} = 0$, we have the conventional Ohms law: $\mathbf{E} = \eta\mathbf{J}$
  - In many plasma application resistivity can be neglected

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = 0$$

    - Thus, there is no parallel electric field, $\mathbf{E}_{\parallel} = 0$
    - When in the rest frame without resistivity, $\mathbf{E} = 0$
- The *mass continuity equation* for the mass density ρ_m:

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\mathbf{v}\rho_m) = 0$$

- …but there are more equations!

Maxwell's equations for MHD

- The plasma moves with velocity $\mathbf{v}$, momentum conservation:

$$\rho_m \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla \mathbf{v})\right] = \mathbf{J} \times \mathbf{B} - \nabla p$$

  - Note: the term $\mathbf{v} \cdot \nabla \mathbf{v}$ is non-linear in $\mathbf{v}$, thus it's negligible for small amplitude waves
- Adiabatic pressure: $pn^{-\gamma} = \mathrm{const}$, where γ = adiabatic index
  - Apply gradient: $n\nabla p = \gamma p \nabla n$
  - Linearize for homogeneous plasmas: $n_0 \nabla p_1 = \gamma p_0 \nabla n_1$
  - Defined equilibrium temperature, T_0, such that $p_0 = n_0 T_0$:

$$\nabla p_1 = \gamma T_0 \nabla n_0$$

- The low frequency assumption in MHD simplify Maxwell's equations!
  - Charge separation is impossible!

$$\nabla \cdot \mathbf{J} = 0 \;\&\; \nabla \cdot \mathbf{E} = 0$$

  - Slow events means that the phase velocity is smaller than c:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Maxwell's equations for MHD

MHD equations		
$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\mathbf{v}\rho_m) = 0$	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	$\nabla \cdot \mathbf{B} = 0$
$\rho_m \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla \mathbf{v})\right] = \mathbf{J} \times \mathbf{B} - \nabla p$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \cdot \mathbf{E} = 0$
$pn^{-\gamma} = \mathrm{const}$	$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}$	$\nabla \cdot \mathbf{J} = 0$

- The MHD equations then includes 4 vector equations and 5 scalar equations; altogether 17 coupled differential equations!!
- Non-linear equations – linearisation often required
- MHD equations are simplified by elimination of variable
  - E.g. eliminate $\mathbf{J}$ using Ampere's law and $\mathbf{E}$ from Ohm's law
- Common simplification, resistivity $\eta = 0$, known as ***ideal MHD***
  - The plasma drift perpendicular to the field lines: $\mathbf{v}_\perp = \mathbf{E} \times \mathbf{B}/B^2$

Transverse waves - Modified light waves

- The waves equation in an unmagnetised plasmas, when $\mathbf{k} = k\mathbf{e}_z$

$$K_{ij} = K(\omega)\delta_{ij} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\delta_{ij}$$

$$\left(n^2\left(\kappa_i\kappa_j - \delta_{ij}\right) + K_{ij}\right)E_j = \begin{bmatrix} K(\omega) - n^2 & 0 & 0 \\ 0 & K(\omega) - n^2 & 0 \\ 0 & 0 & K(\omega) \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Dispersion equation: $(K(\omega) - n^2)^2 K(\omega) = 0$
  - Transverse waves $\left(1 - \frac{\omega_p^2}{\omega^2}\right) - \frac{c^2k^2}{\omega^2} = 0 \Rightarrow \omega^2 - \omega_p^2 - c^2k^2 = 0$
- Dispersion equation: $\omega^2 = c^2k^2 + \omega_p^2$
- These waves are very weakly damped;
  - Phase velocity: $v_{ph}{}^2 = c^2 + \omega_{pe}{}^2/k^2 > c^2$

    thus ***no*** resonant particles and thus ***no Landau damping***!
  - damping can be obtained from collisions;
    for "collision frequency" = ν_e the energy decay rate is: $\gamma_T(k) \approx \nu_e \frac{\omega_{pe}{}^2}{\omega^2}$

Plasma oscillations

- Plasma oscillations: "*the linear reaction of cold and unmagnetised electrons to electrostatic perturbations*"
  - "Cold electrons" = the temperature is negligible.
- Dispersion equation (previous page): $(K(\omega)-n^2)^2K(\omega)=0$
- *Longitudinal* part of dispersion eq.: $K(\omega)=0$

$$1-\frac{\omega_p^2}{\omega^2}=0 \rightarrow \boxed{\omega^2=\omega_p^2}$$

- Note: $v_{gM,i} \equiv \pm \frac{\partial}{\partial k_i}\omega_{pe}=0$

  Thus, plasma oscillation is *not a wave* since no information is propagated by the oscillation!

- However, if we let the electrons have a finite temperature the plasma oscillations are turned into *Langmuir waves*!

Physics of plasma oscillations

- **Model equations**:
  - Electrostatic perturbations follow Poisson's equation

    $$\Delta\phi = \rho/\varepsilon_0$$

    where $\rho = q_i n_i + q_e n_e$ is the charge density.
  - Electron response

    $$m_e \frac{\partial v_e}{\partial t} = q_e \nabla\phi$$

  - Ion response; ions are heavy and do not have time to move: $\mathbf{v}_i = 0$
  - Charge continuity

    $$\frac{\partial\rho}{\partial t} + \nabla\cdot\,\mathbf{J} = 0 \quad \text{, where} \quad \mathbf{J} = q_i n_i \mathbf{v}_i + q_e n_e \mathbf{v}_e$$

Plasma oscillations

- Consider small oscillations near a static equilibrium:

Non-linear (small term)

$$\left.\begin{aligned} \mathbf{v}_e(t) &= 0 + \mathbf{v}_{e1}(t) \\ \phi(t) &= 0 + \phi_1(t) \\ n_e(t) &= n_0 + n_{e1}(t) \\ n_i(t) &= n_0 q_e / q_i + 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \mathbf{J} &= q_e\left(n_{e0}\mathbf{v}_{e1} + n_{e1}\mathbf{v}_{e1}\right) \approx q_e n_{e0}\mathbf{v}_{e1} \\ \rho &= q_e n_{e1} \end{aligned}\right.$$

  – where all the small quantities have sub-index 1.

- Next Fourier transform in time and space

$$\left.\begin{aligned} -k^2\phi_1 &= q_e n_{e1}/\varepsilon_0 \\ -i\omega m_e \mathbf{v}_{e1} &= i q_e \mathbf{k}\phi_1 \\ -i\omega(q_e n_{e1}) + i\mathbf{k}\cdot(q_e n_{e0}\mathbf{v}_{e1}) &= 0 \end{aligned}\right\} \quad \left[\omega^2 - \underbrace{n_{e0}q_e^{\,2}/(\varepsilon_0 m_e)}_{\equiv\,\omega_{pe}^{\,2}}\right] n_{e1} = 0$$

ω_{pe} is the plasma frequency (see previous lecture)

- Dispersion relation: $\boxed{\omega^2 = \omega_{pe}^{\,2}}$

Debye screening

- Debye screening is a static electron response to electrostatic perturbations
- Principle:
  - Electrons tries to screen electrons field
  - But due to thermal motion, fields that are static in rest frame are not static in the frame of a moving electron
  - A moving electron reacts only slow changes, $\omega < \omega_{pe}$
  - A static perturbation will in a moving frame appear as: $\omega = kv$
  - Electrons moving with the thermal speed V_{th} will only screen $k > \omega_{pe}/V_{th}$

Force balance: $0 = -q_e n_{e0} \nabla\phi_1 - T\nabla n_{e,1}$ $\quad \rightarrow n_{e1} = n_{e0} \dfrac{e\phi_1}{T}$

Poisson's eq.: $\varepsilon_0 \nabla \cdot \nabla\phi_1 = -e n_{e,1}$ $\quad \rightarrow \phi_1 = \dfrac{e}{\varepsilon_0 k^2} n_{e,1} = \dfrac{e}{\varepsilon_0 k^2} n_{e0} \dfrac{e}{T}\phi_1$

$$\rightarrow k^2 = \frac{n_{e0} e^2}{\varepsilon_0 T} = \frac{\omega_{pe}^2}{m_e T} = \frac{\omega_{pe}^2}{V_{th}^2}$$

Define the Debye length: $\lambda_D \coloneqq 1/k = \omega_{pe}/V_{th} = n_{e0} e^2/\varepsilon_0 T$

Langmuir waves

- At finite temperature plasma oscillations turn into waves!
- Assume the plasma is isothermal, i.e. the temperature constant

$$\nabla p(t,x) = \gamma T_e \nabla n(t,x)$$

  E.g. in collisionless plasmas.
- The equation of motion, of momentum continuity eq., then reads

$$n_e m_e \frac{\partial \mathbf{v}_e}{\partial t} = -q_e n_{e0} \nabla \phi - \gamma T_e \nabla n_e$$

- Linearise:

$$n_{e0} m_e \frac{\partial \mathbf{v}_{e1}}{\partial t} = -q_e n_{e0} \nabla \phi_1 - \gamma T_e \nabla n_{e1}$$

- Divergence:

$$m_e \frac{\partial}{\partial t} \boxed{n_{e0} \nabla \cdot \mathbf{v}_{e1}} = -q_e n_{e0} \boxed{\nabla \cdot \nabla \phi_1} - \gamma T_e \nabla \cdot \nabla n_{e1}$$

Charge continuity Poisson's equation

$$m_e \frac{\partial}{\partial t} \boxed{\frac{\partial n_{e1}}{\partial t}} = -q_e n_{e0} \boxed{\frac{q_e}{\varepsilon_0} n_{e1}} - \gamma T_e \nabla \cdot \nabla n_{e1}$$

Dispersion equation: $\left(\omega^2 - \omega_{pe}^2 + \gamma V_{th}^2 k^2\right) = 0$

Dispersion relation : $\omega^2 = \omega_{pe}^2 - \gamma V_{th}^2 k^2$

$V_{th}^2 = T_e / m_e$ is the thermal speed

Langmuir waves

- Derivation of Langmuir waves from warm plasma tensor, $\mathbf{k} = k\mathbf{e}_z$

$$K = \begin{pmatrix} K_T & 0 & 0 \\ 0 & K_T & 0 \\ 0 & 0 & K_L \end{pmatrix} \Rightarrow \det\begin{pmatrix} K_T - n^2 & 0 & 0 \\ 0 & K_T - n^2 & 0 \\ 0 & 0 & K_L \end{pmatrix} = 0$$

$$K_L = 1 + \sum_i \frac{1}{k^2\lambda_{Di}^2}\left[1 - \phi(y_i) + i\sqrt{\pi}\,y_i e^{-y_i^2}\right], \quad K_T = \ldots \quad \left\{ \begin{aligned} \lambda_{Di} &= v_{th,i}/\omega_{pi} \\ y_i &= \frac{\omega}{\sqrt{2}kv_{th,i}} \\ v_{th,i} &= \sqrt{T_i/m_i} \end{aligned} \right.$$

- Langmuir wave, the longitudinal solution: $\Re\{K_L\} \approx 0$
- Neglect ions response and expand in small thermal electron velocity (almost cold electrons); use expansion in Eq. (10.30)

$$\phi(y) = 1 + \frac{1}{2y^2} + \frac{3}{4y^4} + \cdots = 1 + \frac{k^2 v_{th,e}^2}{\omega^2} + \frac{3k^4 v_{th,e}^4}{\omega^4} + \cdots$$

$$\omega^2 = \omega_L^2(k) \approx \omega_{pe}^2 + \boxed{3k^2 v_{the}^2}$$

Letting $v_{the}=0$ give plasma oscillations!

Polarization and damping of Langmuir waves

- Polarization vector e_i can be obtained from wave equation when inserting the dispersion relation $K_L \approx K_L^H = 0$

$$\begin{pmatrix} K_T - n^2 & 0 & 0 \\ 0 & K_T - n^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad e_i = \delta_{i3}$$

- Thus, the wave damping can be written as

$$\gamma_L = -2i\omega_L(\mathbf{k})R_L(\mathbf{k})\left\{e_{Li}^*(\mathbf{k})K_{ij}^A\left(\omega_L(\mathbf{k}),\mathbf{k}\right)e_{Lj}(\mathbf{k})\right\} =$$

$$= -2i\omega_L(\mathbf{k})R_L(\mathbf{k})K_{33}^A\left(\omega_L(\mathbf{k}),\mathbf{k}\right) =$$

$$= -2i\omega_L(\mathbf{k})R_L(\mathbf{k})\Im\left\{K_L\left(\omega_L(\mathbf{k}),\mathbf{k}\right)\right\}$$

where $$\frac{1}{R_L(k)} = \omega \left.\frac{\partial \mathrm{Re}\left[K_L(\omega,k)\right]}{\partial \omega}\right|_{\omega=\omega_L(k)}$$

Absorption of Langmuir waves

- Inserting the dispersion relation and the expression for K_L gives the energy dissipation rate

$$\gamma_L \approx \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_{pe}^{\ 4}}{v_{the}^{\ 3} k^3} N_{res} \quad , \text{ where } N_{res} = \exp\left[-v^2/2v_{the}^{\ 2}\right]\Big|_{v=\omega_L(k)/k}$$

  - Damping (dissipation) is due to Landau damping, i.e. for electrons with velocities v such that $\omega_L(k) - kv = 0$
  - Here N_{res} is proportional to the number of Landau resonant electrons
- Damping is small for small & large thermal velocities

$$kv_{the}/\omega_L(k) \to 0 \quad \Rightarrow \quad \gamma_L \sim \lim_{v_{the}\to 0} v_{the}^{\ -3} \exp\left[-v^2/2v_{the}^{\ 2}\right] \to 0$$

$$kv_{the}/\omega_L(k) \to \infty \quad \Rightarrow \quad \gamma_L \sim \lim_{v_{the}\to \infty} v_{the}^{\ -3} \exp\left[-0\right] \to 0$$

  - Maximum in damping is when $v_{the} \approx \omega_L(k)/k$

Ion acoustic waves

- In addition to the Langmuir waves there is another important longitudinal plasma wave (i.e. $K_L=0$) called the ion acoustic wave.
- This mode require motion of both ions and electrons. Assume:
  - Fast electrons: $v_{the} >> \omega/k$, expansions (10.29)
  - Slow ions: $v_{thi} << \omega/k$, expansions (10.30)

$$\Re\{K_L\} = 1 + \underbrace{\frac{1}{k^2\lambda_{De}^2}}_{\text{electron}} - \underbrace{\frac{\omega_{pi}^2}{\omega^2}}_{\text{ion}} \Rightarrow \begin{cases} \omega = \omega_{IA}(k) \approx \dfrac{kv_s}{\sqrt{1+k^2\lambda_{De}^2}} \\ \gamma_L \approx \left(\dfrac{\pi}{2}\right)^{1/2} \omega_{IA}(k)\left(\dfrac{v_s}{v_{the}} + \left(\dfrac{\omega_s(k)}{kv_{the}}\right)^3 N_{res}\right) \end{cases}$$

  - Here v_s is the sounds speed: $v_s = \sqrt{T/m_i}$
  - Again, N_{res} is proportional to the number of Landau resonant electrons
- Ion acoustic waves reduces to normal sounds waves for small $k\lambda_{De}$

$$\omega = \omega_{Sound}(k) \approx kv_s$$

Physics of ion-acoustic waves

- Assume a hydrogen plasma: $q_i = -q_e = e$
- Electrostatic wave; we need Poisson's eq.: $-\varepsilon_0 \nabla \cdot \nabla \phi = en_i - en_e$
- For electrons ion-acoustic waves have low frequency, $\omega \ll \omega_{pe}$; neglect electron mass.

$$0 = -q_e n_{e0} \nabla \phi - T \nabla n_e \rightarrow \boxed{n_{e1} = n_{e0} \frac{e\phi_1}{T}}$$

- Poission eq.: $-\varepsilon_0 \nabla \cdot \nabla \phi_1 + e^2 n_{e0} T^{-1} \phi_1 = en_{i,1}$

$$(k^2 + \lambda_D^{-2})\phi_1 = \frac{en_{i,1}}{\varepsilon_0}$$

- For ions, ion-acoustic waves have low frequency, $\omega \ll \omega_{pi}$;

$$m_i n_i \dot{\mathbf{v}}_i = -e\nabla\phi - \cancel{\gamma_i T \nabla n_i}$$ Not included on previous slide

- Divergence of ion eq. of motion; ion mass continuity & Poission eq.

$$m_i \frac{\partial}{\partial t} n_{i,0} \nabla \cdot \mathbf{v}_{i,1} = -en_{i,0} \nabla \cdot \nabla \phi_1 - T\nabla \cdot \nabla n_{i,1}$$

$$-\omega^2 m_i n_{i,1} = -\frac{e^2 n_{i,0} k^2 n_{i,1}}{\varepsilon_0 (k^2 + \lambda_D^{-2})} - \gamma T k^2 n_{i,1}$$

Ion-acoustic waves

$$\omega^2 = \frac{V_s{}^2 k^2}{\lambda_D^2 k^2 + 1} + \gamma_i V_s{}^2 k^2$$

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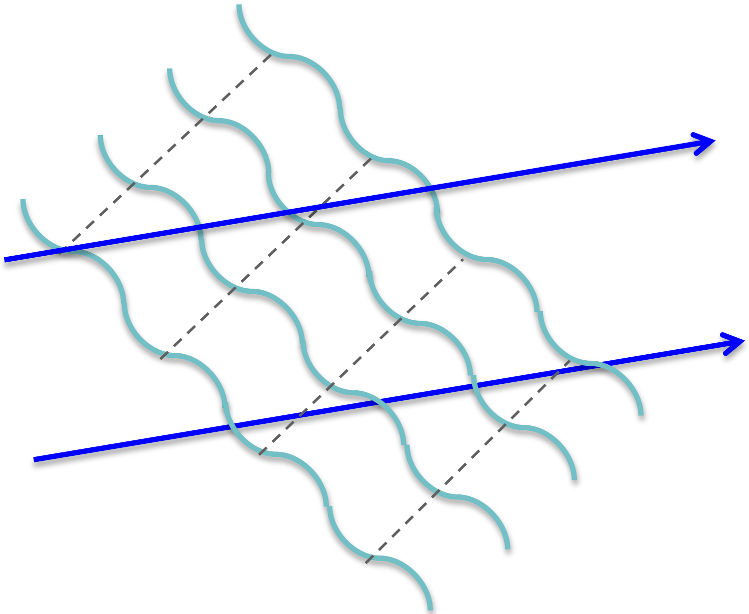


Alfven waves (1)

- Next: Low frequency waves in a cold magnetised plasma including both ions and electrons

- These waves were first studied by Hannes Alfvén, here at KTH in 1940. The wave he discovered is now called the *Alfvén wave*.

- To study these waves we choose:
  $\mathbf{B} \parallel \mathbf{e}_z$ and $\mathbf{k} = (\, k_x \,,\, 0 \,,\, k_{||} \,)$

- The dielectric tensor for these waves were derived in the previous lecture assuming $\omega << \omega_{ci}, \omega_{pi}$

$$K = \begin{pmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & P \end{pmatrix} \qquad \begin{cases} S \approx c^2 \dfrac{\mu_0 \sum_j m_j n_j}{B^2} = \dfrac{c^2}{V_A^{\,2}} \\[2ex] P \approx \dfrac{1}{\omega^2} \sum_j \dfrac{n_j q_j^{\,2}}{m_j \varepsilon_0} = \dfrac{\omega_p^{\,2}}{\omega^2} \end{cases}$$

V_A = "*Alfvén speed*"

Alfven waves (2)

- Wave equation
  - for $n_j = ck_j/\omega$

$$\begin{pmatrix} S - n_{\|}^2 & 0 & -n_{\|}n_x \\ 0 & S - n^2 & 0 \\ -n_{\|}n_x & 0 & P - n_x^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_{\|} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- If you put in numbers, then P is huge!
  - Thus, third equations gives $E_{||} \approx 0$ ($E_{||}$ is the E-field along $\mathbf{B}$)
- Why is $E_{||} \approx 0$ for low frequency waves have?
  - electrons can react very *quickly* to any $E_{||}$ perturbation (along $\mathbf{B}$) and *slowly* to $\mathbf{E}$-perturbations perpendicular to $\mathbf{B}$
  - Thus, they allow E-fields to be perpendicular, but not parallel to $\mathbf{B}$!
- We are then left with a 2D system:

$$\begin{pmatrix} S - n_{\|}^2 & 0 & -- \\ 0 & S - n^2 & -- \\ -- & -- & -- \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Alfven waves (3)

- There are two eigenmodes:

$$\begin{pmatrix} S - n_{\|}^2 & 0 & -- \\ 0 & S - n^2 & -- \\ -- & -- & -- \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \det\left[\Lambda_{ij}\right] = \left(S - n_{\|}^2\right)\left(S - n^2\right) = 0$$

  – The shear Alfvén wave (shear wave): $S = n_{\|}^2$, or $\omega_A(\mathbf{k}) = k_{\|} V_A$

    - Important in almost all areas of plasma physics e.g. fusion plasma stability, space/astrophysical plasmas, molten metals and other laboratory plasmas
    - Polarisation: *see exercise!*

  – The compressional Alfvén wave: $S = n^2$, or $\omega_F(\mathbf{k}) = k V_A$
(fast magnetosonic wave)

    - E.g. used in radio frequency heating of fusion plasmas (my research field)
    - Polarisation: *see exercise!*

Ideal MHD model for Alfven waves

The most simple model that gives the Alfven waves is the linearized *ideal MHD* model for a

- To derive the Alfven wave equation we need four vector equations:

$$nm\frac{d\mathbf{v}}{dt} = \mathbf{j}\times\mathbf{B}_0$$

Momentum balance (sum of electron and ion momentum balance; $n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i = \mathbf{J}$)

$$\mathbf{E} + \mathbf{v}\times\mathbf{B}_0 = \mathbf{0}$$

Ohms law (electron momentum balance when $m_e \rightarrow 0$)

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

Faraday's law

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j}$$

Ampere's law

Wave equation for shear Alfven waves

Derivation of wave equation for the shear wave

1. Substitude $\mathbf{E}$ from Ohms law into Faraday's law

$$\nabla \times (\mathbf{v} \times \mathbf{B}_0) = -\frac{\partial \mathbf{B}}{\partial t}$$

2. Take the time derivative of the equation above and use the momentum balance to eliminate the velocity

$$\nabla \times \left(\left(\frac{\mathbf{j} \times \mathbf{B}_0}{mn} \right) \times \mathbf{B}_0 \right) = -\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

3. Assume the induced current to be perpendicular to $\mathbf{B}_0$

$$\frac{|B_0|^2}{mn} \nabla \times \mathbf{j} = -\frac{\partial^2 \mathbf{B}}{\partial t^2} \qquad \text{Note}: \quad \frac{|B_0|^2}{mn} = \mu_0 V_A{}^2$$

4. Finally use Ampere's law to eliminate $\mathbf{j}$

$$\boxed{\nabla \times (\nabla \times \mathbf{B}) + V_A{}^{-2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0}$$

Wave equation with phase & group velocity V_A

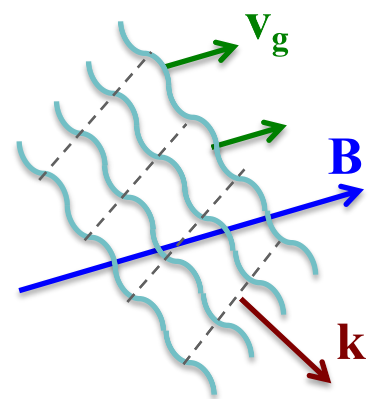
Physics of the shear Alfven waves

- In MHD the plasma is "frozen into the magnetic field" (see course in Plasma Physics)
  - When plasma move, it "pulls" the field line along with it (eq. 1 prev. page)
  - The plasma give the field lines inertia, thus field lines bend back – like guitar strings!
  - Energy transfer during wave motion:
    - **B**-field is *bent* by plasma motion; work needed to bend field line
      - kinetic energy transferred into field line bending
    - Field lines want to unbend and push the plasma back:
      - energy transfer from *field line bending* to kinetic energy
    - … wave motion!

- **B**-field lines acts like strings:
  - The Alfven wave propagates along field lines like waves on a string!
  - Note: the group velocity always points in the direction of the magnetic field, thus it propagate *along* the fields lines!

Group velocities of the shear wave

- Dispersion relation for the shear Alfven wave: $\omega_A(\mathbf{k}) = V_A k_{||} = V_A \mathbf{k} \bullet \mathbf{B} / |\mathbf{B}|$
  - phase velocity: $\mathbf{v}_{phA} \equiv \pm \dfrac{V_A k_{||}}{k} \dfrac{\mathbf{k}}{k}$
  - group velocity: $\mathbf{v}_{gA} \equiv \dfrac{\partial}{\partial \mathbf{k}} \left(\pm V_A k_{||} \right) = \pm V_A \dfrac{\mathbf{B}}{|\mathbf{B}|}$
    - *wave front* moves with $\mathbf{v}_{phA}$, along $\mathbf{k} = (k_x , 0 , k_{||})$
    - wave-energy moves with $\mathbf{v}_{gA}$, along $\mathbf{B} = (0 , 0 , B_0)$!



  - Thus, a shear Alfven waves is "trapped to follow magnetic field lines"
    - like waves propagating along a string
  - Note also:

$$\left| \mathbf{v}_{gA} \right| = V_A \geq \left| \mathbf{v}_{phA} \right|$$

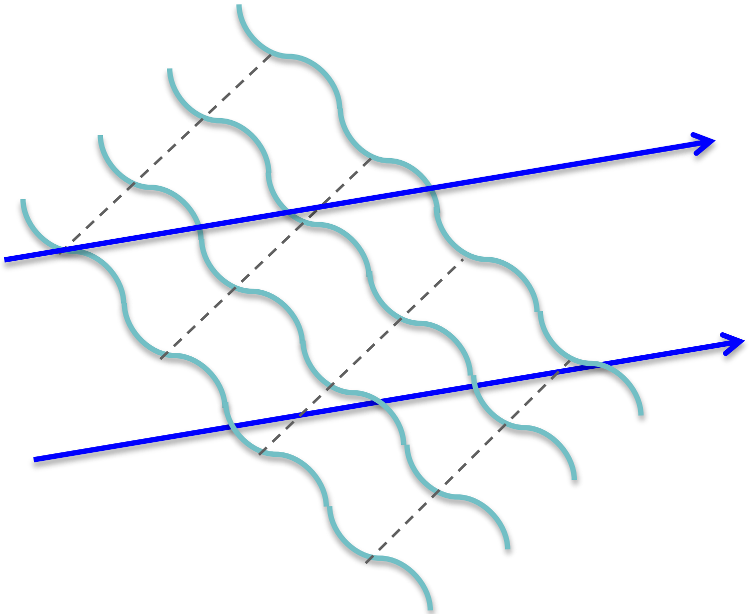
- Fast magnetosonic wave $\omega_F(\mathbf{k}) = V_A k$ is not dispersive!

$$\mathbf{v}_{gF,i} = \mathbf{v}_{phF,i} = V_A \frac{\mathbf{k}}{k}$$

- Thus, an external source may excite two Alfven wave modes propagating in different directions, with different speed

Outline

- Introduction to plasma physics
  - Magneto-Hydro Dynamics, MHD
- Plasmas without magnetic fields
  - Cold plasmas
    - Transverse waves – plasma modified light waves
    - Longitudinal plasma oscillations
  - Warm plasmas – longitudinal waves
    - Debye shilding – incomplete screening at finite temperature
    - Langmuir waves
    - Ion-acoustic (ion-sound) waves
- Alfvén waves - low frequency waves in magnetised plasmas
  - Shear and fast Alfvén waves
- Magnetoionic waves
  - Wave resonances & cut-offs
  - CMA diagram



Magnetoionic theory

- Dielectric response in magnetoionic theory when $\mathbf{B} = B\mathbf{e}_z$

$$K_{ij} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

$$S = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega^2}, D = -\frac{\Omega}{\omega}\frac{\omega_{pe}^2}{\omega^2 - \Omega^2}, P = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

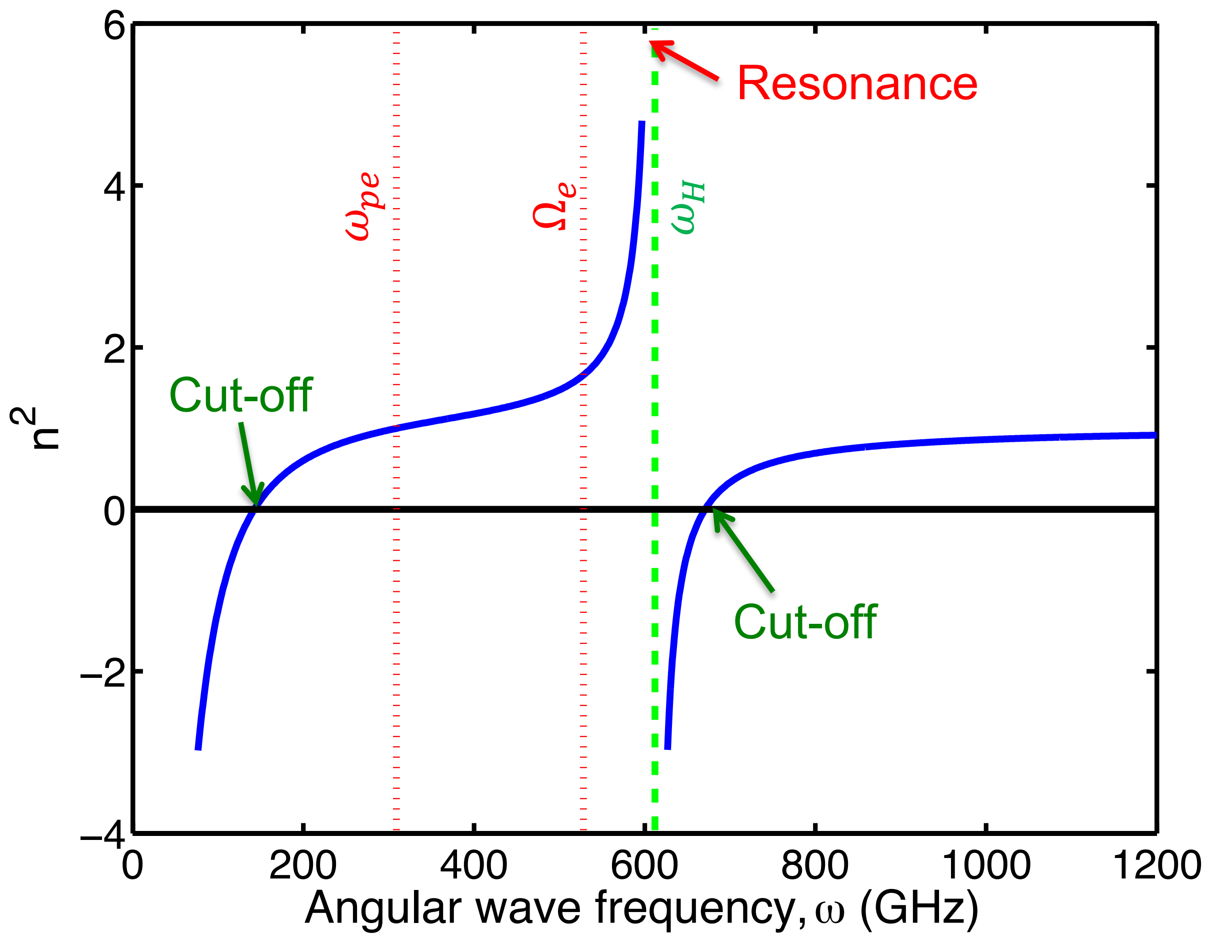
- General solutions are complex, see Melrose & McPhedran.
- Here we'll study $\mathbf{k} = k\mathbf{e}_z$, i.e. perpendicular to $\mathbf{B}$.

$$\Lambda_{ij} = \begin{bmatrix} S & -iD & 0 \\ iD & S - n^2 & 0 \\ 0 & 0 & P - n^2 \end{bmatrix}$$

- Dispersion equation: $(P - n^2)[S(S - n^2) - D^2] = 0$
  - Ordinary mode: $n^2 = P$ *like electron gas!*
  - Extraordinary mode: $n^2 = S + \frac{D^2}{S} = \cdots = 1 - \frac{\omega_{pe}^2(\omega^2 - \omega_{pe}^2)}{\omega^2\,(\omega^2 - \omega_H^2)}$
    - where $\omega_H^2 = \Omega^2 + \omega_{pe}^2$ is called the upper hybrid frequency

Magnetoionic theory

- Magnetoionic dispersion relations have have singularities!
- Such singularities are called wave resonances, here at $\omega = \omega_H$
- In addition we have two places where $n^2 = 0$, called cut-off points.

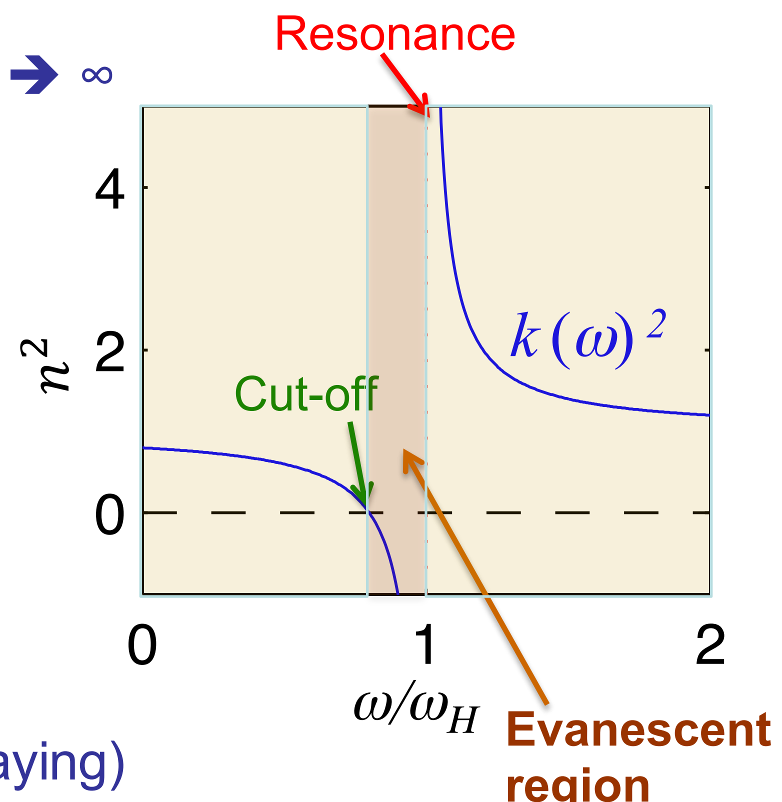


Resonances, cut offs & evanescent waves

- Dispersion relation often has singularities of the form

$$k(\omega)^2 \sim 1 + \frac{\omega_1}{\omega - \omega_{res}} = \frac{\omega - \omega_{cut}}{\omega - \omega_{res}} \quad , \quad \omega_{cut} = \omega_1 - \omega_{res}$$

- The transitions to evanescent waves occur at either
  - ***resonances*** ; k ➔ ∞ , i.e. the wave length λ ➔ 0
    - here at $\omega = \omega_{res}$
  - ***cut-offs*** ; k ➔ 0, i.e. the wave length λ ➔ ∞
    - here at $\omega = \omega_{cut}$



- 3 regions with different types of waves
  - $\omega < \omega_{cut}$ & $\omega > \omega_{res}$, then $k^2 > 0$

$$E \sim E_1 \exp(i|k|x) + E_2 \exp(-i|k|x)$$

  - $\omega_{cut} < \omega < \omega_{res}$, then $k^2 < 0$

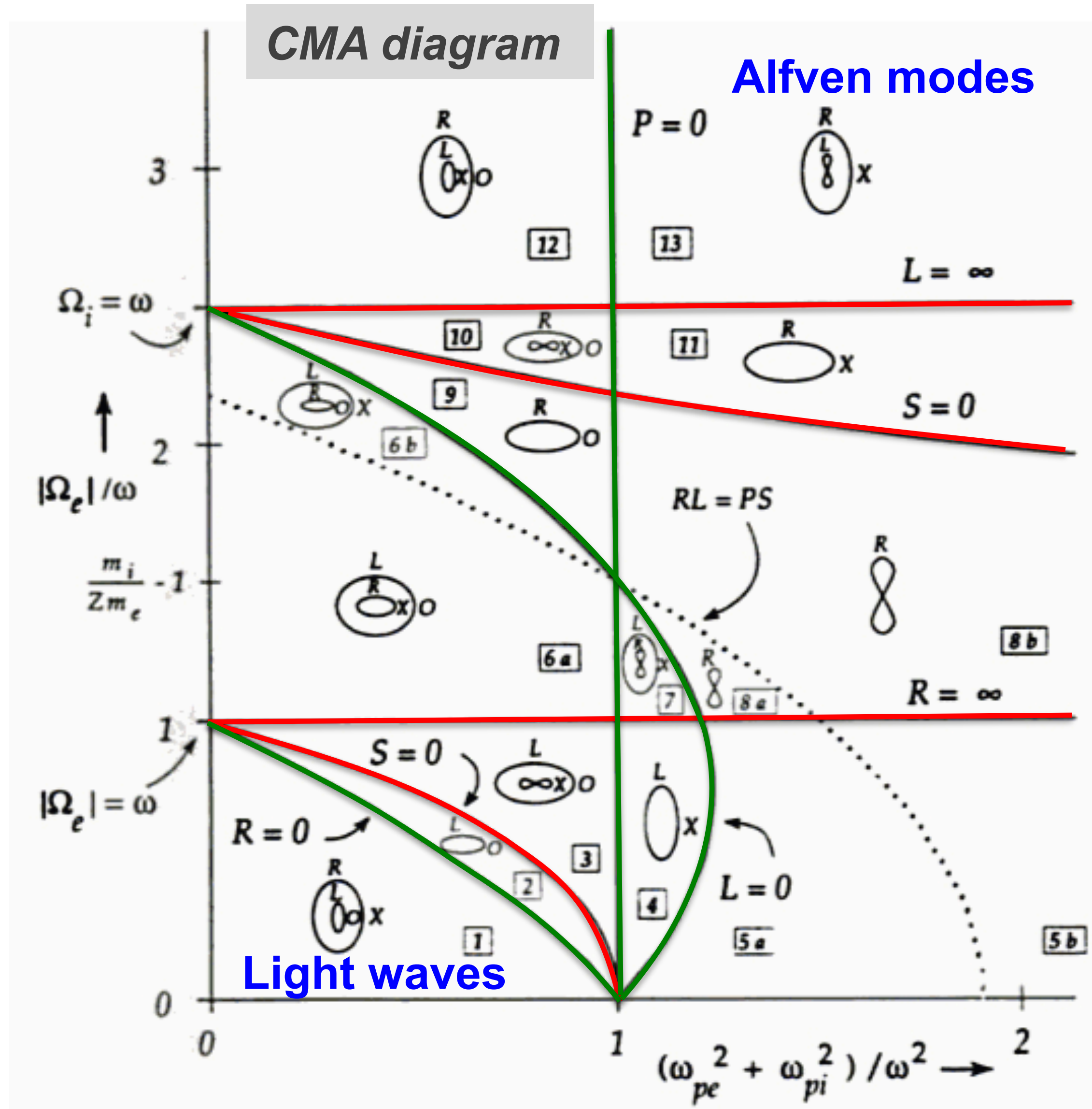
$$E \sim E_1 \exp(|k|x) + E_2 \exp(-|k|x)$$

called **evanescent waves** (growing/decaying)

CMA diagram for cold plasma with ions and electrons ★

- This plasma model can have either 0, 1 or 2 wave modes
- The modes are illustrated in the ***CMA diagram***
- 3 symbols representing different types of anisotropy:
  - ellipse:
  - "eight": 8
  - "infinity": ∞

  (*don't need to know the details*)
- When moving in the diagram mode disappear/appear at:
  - ***resonances*** ; k ➔ ∞
  - ***cut-offs*** ; k ➔ 0



Short summary of plasma waves

- Plasmas are ionised gases – both electromagnetic and acoustic
- Many plasma model – wide range on phenomena
- It is useful to categorise plasma waves by:
  - **Cold / warm** – how do the electrons respond to electron perturbation
    - Cold: screens E-field if perturbation is slower than $1/\omega_{pe}$
    - Warm: cannot screen short wave length $k^{-1} \lesssim \lambda_D$ (in addition to $\omega \gtrsim \omega_{pe}$)
  - **Magnetised / unmagnetised**:
    - Response becomes gyrotropic; add characteristic frequencies Ω_i and Ω_e
    - Alfven wave: shear waves (like strings) & fast compressional waves
  - **High / low** ω : compared to ω_p, Ω_e and Ω_i.
    - For magnetised plasmas, waves in $\Omega_i < \omega < \Omega_e, \omega_p$ are not treated here.

	High freq.	Low freq.
Cold, $\mathbf{B} = 0$	Modified light waves	*No waves!*
Warm, $\mathbf{B} = 0$	Mod. light and Langmuir waves	Ion-acoustic waves
Cold, $\mathbf{B} \neq 0$	Magnetoionic & cold plasma	Cold plasma & MHD: Alfven waves
Warm, $\mathbf{B} \neq 0$	*Not in this course*	*Not in this course*