Waves in plasmas

T. Johnson
Outline

• Introduction to plasma physics
  – Magneto-Hydro Dynamics, MHD

• Plasmas without magnetic fields
  – Cold plasmas
    • Transverse waves – plasma modified light waves
    • Longitudinal plasma oscillations
  – Warm plasmas – longitudinal waves
    • Debye shielding – incomplete screening at finite temperature
    • Langmuir waves
    • Ion-acoustic (ion-sound) waves

• Alfvén waves - low frequency waves in magnetised plasmas
  – Shear and fast Alfvén waves

• Magnetoionic waves
  – Wave resonances & cut-offs
  – CMA diagram
Introduction of plasma

- Plasmas are ionised gases
- High temperature and low concentration; use Newtonian mechanics
  \[ m\dot{v} = q(E + v \times B) \]
- High conductivity
  - difficult to produce charge separation – plasmas are quasineutral;
  - except at high frequency, \( \omega > \omega_{pe} \), electrons too heavy to react
- Magnetic fields: cause particles to follow gyro orbits
  - \( v \times B \) force cause particles to gyrate around the magnetic field lines.
  - Gyro frequency: \( \Omega = qB/m \)
  - Gyro radius: \( \rho = v/\Omega = mv/qB \)
  - Plasmas can be very anisotropic, in fusion plasmas:
    \[ \frac{\sigma_{||}}{\sigma_{\perp}} \sim 10^9 \]
    i.e. it is almost impossible to conduct currents perpendicular to \( B \)!
Plasma models

• There are many mathematical plasma models
• We have the cold plasma models, i.e. without thermal motion
  – Magneto-ionic theory (inc. electron response, while ions are static)
  – Cold plasma – inc. both electron and ion responses

• And the general warm plasma model from lecture 5.
  – This one can also be generalised to include magnetised plasmas.

• Many of these models are too complicated to analyse and solve
  – Almost all practical solutions involve further approximations
  – Typically: expand for a specific range of frequencies, wave lengths, or phase velocity
  – Example:
    \[ \omega \ll \omega_{pi}, \Omega_i, V_{th}k \ll \Omega_i \ldots \]
    These are the main conditions for the so called magneto-hydro dynamics model
The MHD model for a plasma

- Magneto-Hydro Dynamics (MHD), the most famous plasma model!
- It assumes low frequencies compared to the ion cyclotron frequency, \( \omega \ll \Omega_i \) and the plasma frequency, \( \omega \ll \omega_{pe} \).
- **Assumption:** electrons have an infinite small mass:
  \[
  \mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}
  \]
  This equation is referred to as *Ohms law* and \( \eta \) is the resistivity.
  - In the rest frame, \( \mathbf{v} = 0 \), we have the conventional Ohms law: \( \mathbf{E} = \eta \mathbf{J} \)
  - In many plasma application resistivity can be neglected
    \[
    \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0
    \]
    - Thus, there is no parallel electric field, \( \mathbf{E}_\parallel = 0 \)
    - When in the rest frame without resistivity, \( \mathbf{E} = 0 \)

- The *mass continuity equation* for the mass density \( \rho_m \):
  \[
  \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\mathbf{v} \rho_m) = 0
  \]
- …but there are more equations!
Maxwell’s equations for MHD

• The plasma moves with velocity \( \mathbf{v} \), momentum conservation:
  \[
  \rho_m \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla \mathbf{v}) \right] = \mathbf{J} \times \mathbf{B} - \nabla p
  \]
  – Note: the term \( \mathbf{v} \cdot \nabla \mathbf{v} \) is non-linear in \( \mathbf{v} \), thus it’s negligible for small amplitude waves

• Adiabatic pressure: \( pn^{-\gamma} = \text{const} \), where \( \gamma = \) adiabatic index
  – Apply gradient: \( n\nabla p = \gamma p\nabla n \)
  – Linearize for homogeneous plasmas: \( n_0\nabla p_1 = \gamma p_0\nabla n_1 \)
  – Defined equilibrium temperature, \( T_0 \), such that \( p_0 = n_0T_0 \):
    \[
    \nabla p_1 = \gamma T_0 \nabla n_0
    \]

• The low frequency assumption in MHD simplify Maxwell’s equations!
  – Charge separation is impossible!
    \[
    \nabla \cdot \mathbf{J} = 0 \quad \text{&} \quad \nabla \cdot \mathbf{E} = 0
    \]
  – Slow events means that the phase velocity is smaller than \( c \):
    \[
    \nabla \times \mathbf{B} = \mu_0 \mathbf{J}
    \]
Maxwell’s equations for MHD

<table>
<thead>
<tr>
<th>MHD equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\mathbf{v} \rho_m) = 0 )</td>
</tr>
<tr>
<td>( \rho_m \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \mathbf{J} \times \mathbf{B} - \nabla p )</td>
</tr>
<tr>
<td>( \rho n^{\nu} = \text{const} )</td>
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</tbody>
</table>

- The MHD equations then includes 4 vector equations and 5 scalar equations; altogether 17 coupled differential equations!!
- Non-linear equations – linearisation often required
- MHD equations are simplified by elimination of variable
  - E.g. eliminate \( \mathbf{J} \) using Ampere’s law and \( \mathbf{E} \) from Ohm’s law
- Common simplification, resistivity \( \eta = 0 \), known as ideal MHD
  - The plasma drift perpendicular to the field lines: \( \mathbf{v}_\perp = \mathbf{E} \times \mathbf{B} / B^2 \)
Transverse waves - Modified light waves

• The waves equation in an unmagnetised plasmas, when $\mathbf{k} = k \mathbf{e}_z$

$$K_{ij} = K(\omega) \delta_{ij} = \left(1 - \frac{\omega_p^2}{\omega^2}\right) \delta_{ij}$$

$$(n^2(\kappa_i \kappa_j - \delta_{ij}) + K_{ij})E_j = K(\omega) - n^2 0 0$$

• Dispersion equation: $(K(\omega) - n^2)^2 K(\omega) = 0$

  – Transverse waves

$$\left(1 - \frac{\omega_p^2}{\omega^2}\right) - \frac{c^2 k^2}{\omega^2} = 0 \quad \Rightarrow \quad \omega^2 - \omega_p^2 - c^2 k^2 = 0$$

• Dispersion equation: $\omega^2 = c^2 k^2 + \omega_p^2$

• These waves are very weakly damped;

  – Phase velocity: $\nu_{ph}^2 = c^2 + \omega_{pe}^2 / k^2 > c^2$

    thus no resonant particles and thus no Landau damping!

  – damping can be obtained from collisions;

    for “collision frequency” = $\nu_e$ the energy decay rate is: $\gamma_T(k) \approx \nu_e \frac{\omega_{pe}^2}{\omega^2}$

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Dispersive Media, Lecture 5 - Thomas Johnson
Plasma oscillations

- **Plasma oscillations**: “the linear reaction of cold and unmagnetised electrons to electrostatic perturbations”
  - “Cold electrons” = the temperature is negligible.

- Dispersion equation (previous page):
  \[ (K(\omega) - n^2)^2 K(\omega) = 0 \]

- *Longitudinal* part of dispersion eq.: \[ K(\omega) = 0 \]
  \[
  1 - \frac{\omega_p^2}{\omega^2} = 0 \quad \rightarrow \quad \omega^2 = \omega_p^2
  \]

- Note: \[ v_{gM,i} \equiv \pm \frac{\partial}{\partial k_i} \omega_{pe} = 0 \]

  Thus, plasma oscillation is *not a wave* since no information is propagated by the oscillation!

- However, if we let the electrons have a finite temperature the plasma oscillations are turned into *Langmuir waves*!
Physics of plasma oscillations

- Model equations:
  - Electrostatic perturbations follow Poisson’s equation
    \[ \Delta \phi = \rho / \varepsilon_0 \]
    where \( \rho = q_i n_i + q_e n_e \) is the charge density.
  - Electron response
    \[ m_e \frac{\partial v_e}{\partial t} = q_e \nabla \phi \]
  - Ion response; ions are heavy and do not have time to move: \( v_i = 0 \)
  - Charge continuity
    \[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \]
    , where \( \mathbf{J} = q_i n_i \mathbf{v}_i + q_e n_e \mathbf{v}_e \)
Plasma oscillations

• Consider small oscillations near a static equilibrium:

\[
\begin{align*}
\mathbf{v}_e(t) &= 0 + \mathbf{v}_{e1}(t) \\
\phi(t) &= 0 + \phi_1(t) \\
n_e(t) &= n_0 + n_{e1}(t) \\
n_i(t) &= n_0 q_e / q_i + 0
\end{align*}
\]

– where all the small quantities have sub-index 1.

• Next Fourier transform in time and space

\[
\begin{align*}
-k^2 \phi_1 &= q_e n_{e1} / \varepsilon_0 \\
-\imath \omega m_e \mathbf{v}_{e1} &= \imath q_e \mathbf{k} \phi_1 \\
-\imath \omega (q_e n_{e1}) + \imath \mathbf{k} \cdot (q_e n_{e0} \mathbf{v}_{e1}) &= 0
\end{align*}
\]

\[
\left[ \omega^2 - \frac{n_{e0} q_e^2}{(\varepsilon_0 m_e)} \right] n_{e1} = 0
\]

\[
\equiv \omega_{pe}^2
\]

ω_{pe} is the plasma frequency (see previous lecture)

• Dispersion relation:

\[
\omega^2 = \omega_{pe}^2
\]
Debye screening

• Debye screening is a static electron response to electrostatic perturbations
• Principle:
  – Electrons tries to screen electrons field
  – But due to thermal motion, fields that are static in rest frame are not static in the frame of a moving electron
  – A moving electron reacts only slow changes, $\omega < \omega_{pe}$
  – A static perturbation will in a moving frame appear as: $\omega = kv$
  – Electrons moving with the thermal speed $V_{th}$ will only screen $k > \omega_{pe}/V_{th}$

Force balance: $0 = -q_e n_e \nabla \phi_1 - T \nabla n_{e,1}$ \hspace{1cm} $\rightarrow n_{e1} = n_{e0} \frac{e \phi_1}{T}$

Poisson’s eq.: $\varepsilon_0 \nabla \cdot \nabla \phi_1 = -en_{e,1}$ \hspace{1cm} $\rightarrow \phi_1 = \frac{e}{\varepsilon_0 k^2} n_{e,1} = \frac{e}{\varepsilon_0 k^2} n_{e0} \frac{e}{T} \phi_1$

$\rightarrow k^2 = \frac{n_{e0} e^2}{\varepsilon_0 T} = \frac{\omega_{pe}^2}{m_e T} = \frac{\omega_{pe}^2}{V_{th}^2}$

Define the Debye length: $\lambda_D := 1/k = \omega_{pe}/V_{th} = n_{e0} e^2 / \varepsilon_0 T$
Langmuir waves

• At finite temperature plasma oscillations turn into waves!
• Assume the plasma is isothermal, i.e. the temperature constant
  \[ \nabla p(t, x) = \gamma T_e \nabla n(t, x) \]
  E.g. in collisionless plasmas.
• The equation of motion, of momentum continuity eq., then reads
  \[ n_e m_e \frac{\partial \mathbf{v}_e}{\partial t} = -q_e n_e \nabla \phi - \gamma T_e \nabla n_e \]
• Linearise:
  \[ n_0 m_e \frac{\partial \mathbf{v}_{e1}}{\partial t} = -q_e n_e \nabla \phi_1 - \gamma T_e \nabla n_{e1} \]
• Divergence:
  \[ m_e \frac{\partial}{\partial t} n_e \nabla \cdot \mathbf{v}_{e1} = -q_e n_e \nabla \cdot \nabla \phi_1 - \gamma T_e \nabla \cdot \nabla n_{e1} \]
  Charge continuity  Poisson’s equation
  \[ m_e \frac{\partial}{\partial t} \frac{\partial n_{e1}}{\partial t} = -q_e n_e \frac{q_e}{\varepsilon_0} n_{e1} - \gamma T_e \nabla \cdot \nabla n_{e1} \]
  Dispersion equation: \( (\omega^2 - \omega_{pe}^2 + \gamma V_{th}^2 k^2) = 0 \)
  \( V_{th}^2 = T_e/m_e \) is the thermal speed
  Dispersion relation: \( \omega^2 = \omega_{pe}^2 - \gamma V_{th}^2 k^2 \)
Langmuir waves

- Derivation of Langmuir waves from warm plasma tensor, \( \mathbf{k} = k \mathbf{e}_z \)

\[
K = \begin{pmatrix}
  K_T & 0 & 0 \\
  0 & K_T & 0 \\
  0 & 0 & K_L
\end{pmatrix}
\rightarrow \det \begin{pmatrix}
  K_T - n^2 & 0 & 0 \\
  0 & K_T - n^2 & 0 \\
  0 & 0 & K_L
\end{pmatrix} = 0
\]

\[
K_L = 1 + \sum_i \frac{1}{k^2 \lambda^2_{Di}} \left[ 1 - \phi(y_i) + i \sqrt{\pi} y_i e^{-y_i^2} \right], \quad K_T = \ldots
\]

\[
\lambda_{Di} = \frac{v_{th,i}}{\omega_p} \\
y_i = \frac{\omega}{\sqrt{2 k v_{th,i}}} \\
v_{th,i} = \sqrt{T_i/m_i}
\]

- Langmuir wave, the longitudinal solution: \( \Re \{ K_L \} \approx 0 \)

- Neglect ions response and expand in small thermal electron velocity (almost cold electrons); use expansion in Eq. (10.30)

\[
\phi(y) = 1 + \frac{1}{2 y^2} + \frac{3}{4 y^4} + \cdots = 1 + \frac{k^2 v_{th,e}^2}{\omega^2} + \frac{3 k^4 v_{th,e}^4}{\omega^4} + \cdots
\]

\[
\omega^2 = \omega_L^2(k) \approx \omega_{pe}^2 + 3k^2 v_{the}^2
\]

Letting \( v_{the} = 0 \) give plasma oscillations!
Polarization and damping of Langmuir waves

- Polarization vector $e_i$ can be obtained from wave equation when inserting the dispersion relation $K_L \approx K_L^H = 0$

\[
\begin{pmatrix}
K_T - n^2 & 0 & 0 \\
0 & K_T - n^2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix} = \begin{pmatrix}0 \\
0 \\
1
\end{pmatrix} \quad \Rightarrow \quad e_i = \delta_{i3}
\]

- Thus, the wave damping can be written as

\[
\gamma_L = -2i\omega_L(k)R_L(k) \left\{ e_{Li}^*(k)K_{ij}^A(\omega_L(k),k)e_{Lj}(k) \right\} =
\]

\[
= -2i\omega_L(k)R_L(k)K_{33}^A(\omega_L(k),k) =
\]

\[
= -2i\omega_L(k)R_L(k)\Im\left\{ K_L(\omega_L(k),k) \right\}
\]

where

\[
\frac{1}{R_L(k)} = \omega \frac{\partial \text{Re} \left[ K_L(\omega,k) \right]}{\partial \omega} \bigg|_{\omega = \omega_L(k)}
\]
Absorption of Langmuir waves

- Inserting the dispersion relation and the expression for $K_L$ gives the energy dissipation rate

\[ \gamma_L \approx \left( \frac{\pi}{2} \right)^{1/2} \frac{\omega_{pe}^4}{v_{the}^3 k^3} N_{res} \], where \( N_{res} = \exp\left[-v^2/2v_{the}^2 \right] \)

- Damping (dissipation) is due to Landau damping, i.e. for electrons with velocities $v$ such that $\omega_L(k) - kv = 0$

- Here $N_{res}$ is proportional to the number of Landau resonant electrons

- Damping is small for small & large thermal velocities

\[ kv_{the}/\omega_L(k) \to 0 \quad \to \quad \gamma_L \sim \lim_{v_{the} \to 0} v_{the}^{-3} \exp\left[-v^2/2v_{the}^2 \right] \to 0 \]

\[ kv_{the}/\omega_L(k) \to \infty \quad \to \quad \gamma_L \sim \lim_{v_{the} \to \infty} v_{the}^{-3} \exp\left[-0 \right] \to 0 \]

- Maximum in damping is when $v_{the} \approx \omega_L(k)/k$
Ion acoustic waves

- In addition to the Langmuir waves there is another important longitudinal plasma wave (i.e. $K_L = 0$) called the ion acoustic wave.
- This mode require motion of both ions and electrons. Assume:
  - Fast electrons: $v_{the} \gg \omega/k$, expansions (10.29)
  - Slow ions: $v_{thi} \ll \omega/k$, expansions (10.30)

\[
\mathcal{R}\{K_L\} = 1 \pm \frac{1}{k^2 \lambda_{De}^2} - \frac{\omega_{pi}^2}{\omega^2} \quad \omega = \omega_{IA}(k) \approx \frac{k v_s}{\sqrt{1 + k^2 \lambda_{De}^2}}
\]

\[
\gamma_L \approx \left(\frac{\pi}{2}\right)^{1/2} \omega_{IA}(k) \left(\frac{v_s}{v_{the}} + \left(\frac{\omega_{s}(k)}{k v_{the}}\right)^3 N_{res}\right)
\]

- Here $v_s$ is the sounds speed: $v_s = \sqrt{T/m_i}$
- Again, $N_{res}$ is proportional to the number of Landau resonant electrons
- Ion acoustic waves reduces to normal sounds waves for small $k \lambda_{De}$

\[
\omega = \omega_{\text{Sound}}(k) \approx k v_s
\]
Physics of ion-acoustic waves

- Assume a hydrogen plasma: \( q_i = -q_e = e \)
- Electrostatic wave; we need Poisson’s eq.: \( -\varepsilon_0 \nabla \cdot \nabla \phi = e n_i - en_e \)
- For electrons ion-acoustic waves have low frequency, \( \omega \ll \omega_{pe} \); neglect electron mass.

\[
0 = -q_e n_e \phi - T \nabla n_e \rightarrow n_{e1} = n_e \frac{e \phi_1}{T}
\]

- Poisson eq.: \( -\varepsilon_0 \nabla \cdot \nabla \phi_1 + e^2 n_e T^{-1} \phi_1 = e n_{i,1} \)

\[
(k^2 + \lambda_D^{-2}) \phi_1 = \frac{en_{i,1}}{\varepsilon_0}
\]

- For ions, ion-acoustic waves have low frequency, \( \omega \ll \omega_{pi} \);

\[
m_i n_i \ddot{v}_i = -e \nabla \phi - \gamma_i T \nabla n_i 
\]

- Divergence of ion eq. of motion; ion mass continuity & Poission eq.

\[
m_i \frac{\partial}{\partial t} n_{i,0} \nabla \cdot v_{i,1} = -en_{i,0} \nabla \cdot \nabla \phi_1 - T \nabla \cdot \nabla n_{i,1}
\]

\[
-\omega^2 m_i n_{i,1} = -\frac{e^2 n_{i,0} k^2 n_{i,1}}{\varepsilon_0 (k^2 + \lambda_D^{-2})} - \gamma T k^2 n_{i,1}
\]

\[
\omega^2 = \frac{V_s^2 k^2}{\lambda_D^2 k^2 + 1} + \gamma_i V_s^2 k^2
\]

Ion-acoustic waves

Not included on previous slide
• Introduction to plasma physics
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  – Warm plasmas – longitudinal waves
    • Debye shielding – incomplete screening at finite temperature
    • Langmuir waves
    • Ion-acoustic (ion-sound) waves
• Alfvén waves - low frequency waves in magnetised plasmas
  – Shear and fast Alfvén waves
• Magnetoionic waves
  – Wave resonances & cut-offs
  – CMA diagram
Alfven waves (1)

- Next: Low frequency waves in a cold magnetised plasma including both ions and electrons

- These waves were first studied by Hannes Alfvén, here at KTH in 1940. The wave he discovered is now called the Alfvén wave.

To study these waves we choose:

\[ \mathbf{B} \parallel \mathbf{e}_z \text{ and } \mathbf{k} = (k_x, 0, k_\parallel) \]

- The dielectric tensor for these waves were derived in the previous lecture assuming \( \omega \ll \omega_{ci}, \omega_{pi} \)

\[
\begin{pmatrix}
S & 0 & 0 \\
0 & S & 0 \\
0 & 0 & P
\end{pmatrix}
\]

\[
S \approx c^2 \frac{\mu_0 \sum_j m_j n_j}{B^2} = \frac{c^2}{V_A^2}
\]

\[ V_A = "\text{Alfvén speed}" \]

\[
P \approx \frac{1}{\omega^2} \sum_j \frac{n_j q_j^2}{m_j \varepsilon_0} = \frac{\omega_p^2}{\omega^2}
\]
Alfvén waves (2)

- Wave equation
  - for \( n_j = \frac{c k_j}{\omega} \)
    \[
    \begin{pmatrix}
    S - n_{||}^2 & 0 & -n_{||} n_x \\
    0 & S - n^2 & 0 \\
    -n_{||} n_x & 0 & P - n_x^2
    \end{pmatrix}
    \begin{pmatrix}
    E_x \\
    E_y \\
    E_{||}
    \end{pmatrix}
    = \begin{pmatrix}
    0 \\
    0 \\
    0
    \end{pmatrix}
    \]

- If you put in numbers, then \( P \) is huge!
  - Thus, third equations gives \( E_{||} \approx 0 \) (\( E_{||} \) is the E-field along \( B \))

- Why is \( E_{||} \approx 0 \) for low frequency waves have?
  - electrons can react very quickly to any \( E_{||} \) perturbation (along \( B \)) and slowly to \( E \)-perturbations perpendicular to \( B \)
  - Thus, they allow \( E \)-fields to be perpendicular, but not parallel to \( B \)!

- We are then left with a 2D system:
  \[
  \begin{pmatrix}
  S - n_{||}^2 & 0 \\
  0 & S - n^2 \\
  -n_{||} n_x & 0
  \end{pmatrix}
  \begin{pmatrix}
  E_x \\
  E_y \\
  0
  \end{pmatrix}
  = \begin{pmatrix}
  0 \\
  0 \\
  0
  \end{pmatrix}
  \]
Alfvén waves (3)

- There are two eigenmodes:

\[
\begin{pmatrix}
S - n_{\parallel}^2 & 0 \\
0 & S - n^2
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[\det[\Lambda_{ij}] = (S - n_{\parallel}^2)(S - n^2) = 0\]

- The shear Alfvén wave (shear wave): \( S = n_{\parallel}^2 \), or \( \omega_A(\mathbf{k}) = k_{\parallel}V_A \)

  - Important in almost all areas of plasma physics e.g. fusion plasma stability, space/astrophysical plasmas, molten metals and other laboratory plasmas
  - Polarisation: see exercise!

- The compressional Alfvén wave: \( S = n^2 \), or \( \omega_F(\mathbf{k}) = kV_A \)

  - E.g. used in radio frequency heating of fusion plasmas (my research field)
  - Polarisation: see exercise!
Ideal MHD model for Alfven waves

The most simple model that gives the Alfven waves is the linearized *ideal MHD* model for a

- To derive the Alfven wave equation we need four vector equations:

\[
 nm \frac{dv}{dt} = j \times B_0 \quad \text{Momentum balance} \\
 (\text{sum of electron and ion momentum balance;} \quad n_e q_e v_e + n_i q_i v_i = J)
\]

\[
 E + v \times B_0 = 0 \quad \text{Ohms law} \\
 (\text{electron momentum balance when} \ m_e \to 0)
\]

\[
 \nabla \times E = -\frac{\partial B}{\partial t} \quad \text{Faraday’s law}
\]

\[
 \nabla \times B = \mu_0 j \quad \text{Ampere’s law}
\]
Derivation of wave equation for the shear wave

1. Substitute $\mathbf{E}$ from Ohms law into Faraday’s law

$$\nabla \times (\mathbf{v} \times \mathbf{B}_0) = -\frac{\partial \mathbf{B}}{\partial t}$$

2. Take the time derivative of the equation above and use the momentum balance to eliminate the velocity

$$\nabla \times \left(\left(\frac{\mathbf{j} \times \mathbf{B}_0}{mn}\right) \times \mathbf{B}_0\right) = -\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

3. Assume the induced current to be perpendicular to $\mathbf{B}_0$

$$\frac{|\mathbf{B}_0|^2}{mn} \nabla \times \mathbf{j} = -\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Note: $\frac{|\mathbf{B}_0|^2}{mn} = \mu_0 V_A^2$

4. Finally use Ampere’s law to eliminate $\mathbf{j}$

$$\nabla \times (\nabla \times \mathbf{B}) + V_A^{-2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

Wave equation with phase & group velocity $V_A$
Physics of the shear Alfven waves

• In MHD the plasma is “frozen into the magnetic field” (see course in Plasma Physics)
  – When plasma move, it “pulls” the field line along with it (eq. 1 prev. page)
  – The plasma give the field lines inertia, thus field lines bend back – like guitar strings!
  – Energy transfer during wave motion:
    • $B$-field is bent by plasma motion; work needed to bend field line
      – kinetic energy transferred into field line bending
    • Field lines want to unbend and push the plasma back:
      – energy transfer from field line bending to kinetic energy
    • … wave motion!

• $B$-field lines acts like strings:
  – The Alfven wave propagates along field lines like waves on a string!
  – Note: the group velocity always points in the direction of the magnetic field, thus it propagate along the fields lines!
Group velocities of the shear wave

- Dispersion relation for the shear Alfven wave: $\omega_A(k) = V_A k_\parallel = V_A \mathbf{k} \cdot \mathbf{B} / |\mathbf{B}|$
  
  - phase velocity: $v_{phA} = \pm \frac{V_A k_\parallel}{k} \frac{k}{k}$
  
  - group velocity: $v_{gA} = \frac{\partial}{\partial \mathbf{k}} (\pm V_A k_\parallel) = \pm V_A \frac{\mathbf{B}}{|\mathbf{B}|}$
    
    - wave front moves with $v_{phA}$, along $\mathbf{k} = (k_x, 0, k_\parallel)$
    - wave-energy moves with $v_{gA}$, along $\mathbf{B} = (0, 0, B_0)$!
  
  - Thus, a shear Alfven waves is “trapped to follow magnetic field lines”
    
    - like waves propagating along a string
  
  - Note also:
    
    $|v_{gA}| = V_A \geq |v_{phA}|$
  
- Fast magnetosonic wave $\omega_F(k) = V_A k$ is not dispersive!
  
  $v_{gF,i} = v_{phF,i} = V_A \frac{k}{k}$
  
  - Thus, an external source may excite two Alfven wave modes propagating in different directions, with different speed
Outline

- Introduction to plasma physics
  - Magneto-Hydro Dynamics, MHD
- Plasmas without magnetic fields
  - Cold plasmas
    - Transverse waves – plasma modified light waves
    - Longitudinal plasma oscillations
  - Warm plasmas – longitudinal waves
    - Debye shielding – incomplete screening at finite temperature
    - Langmuir waves
    - Ion-acoustic (ion-sound) waves
- Alfvén waves - low frequency waves in magnetised plasmas
  - Shear and fast Alfvén waves
- Magnetoionic waves
  - Wave resonances & cut-offs
  - CMA diagram
Magnetoionic theory

- Dielectric response in magnetoionic theory when $\mathbf{B} = B\mathbf{e}_z$
  
  $$K_{ij} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

  $$S = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega^2}, D = -\frac{\Omega}{\omega} \frac{\omega_{pe}^2}{\omega^2 - \Omega^2}, P = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

- General solutions are complex, see Melrose & McPhedran.

- Here we’ll study $\mathbf{k} = k\mathbf{e}_z$, i.e. perpendicular to $\mathbf{B}$.

  $$\Lambda_{ij} = \begin{bmatrix} S & -iD & 0 \\ iD & S - n^2 & 0 \\ 0 & 0 & P - n^2 \end{bmatrix}$$

- Dispersion equation: $(P - n^2)[S(S - n^2) - D^2] = 0$
  - Ordinary mode: $n^2 = P$  like electron gas!
  - Extraordinary mode: $n^2 = S + \frac{D^2}{S} = \cdots = 1 - \frac{\omega_{pe}^2 (\omega^2 - \omega_{pe}^2)}{\omega^2 (\omega^2 - \omega_H^2)}$

  - where $\omega_H^2 = \Omega^2 + \omega_{pe}^2$ is called the upper hybrid frequency
Magnetoionic theory

- Magnetoionic dispersion relations have singularities!
- Such singularities are called wave resonances, here at $\omega = \omega_H$
- In addition we have two places where $n^2 = 0$, called cut-off points.
Resonances, cut offs & evanescent waves

- Dispersion relation often has singularities of the form

\[
k(\omega)^2 \approx 1 + \frac{\omega_1}{\omega - \omega_{\text{res}}} = \frac{\omega - \omega_{\text{cut}}}{\omega - \omega_{\text{res}}}, \quad \omega_{\text{cut}} = \omega_1 - \omega_{\text{res}}
\]

- The transitions to evanescent waves occur at either
  - **resonances**; \( k \rightarrow \infty \), i.e. the wave length \( \lambda \rightarrow 0 \)
    - here at \( \omega = \omega_{\text{res}} \)
  - **cut-offs**; \( k \rightarrow 0 \), i.e. the wave length \( \lambda \rightarrow \infty \)
    - here at \( \omega = \omega_{\text{cut}} \)

- 3 regions with different types of waves
  - \( \omega < \omega_{\text{cut}} \) & \( \omega > \omega_{\text{res}} \), then \( k^2 > 0 \)
    \[
    E \sim E_1 \exp(i|k|x) + E_2 \exp(-i|k|x)
    \]
  - \( \omega_{\text{cut}} < \omega < \omega_{\text{res}} \), then \( k^2 < 0 \)
    \[
    E \sim E_1 \exp(|k|x) + E_2 \exp(-|k|x)
    \]
    called **evanescent waves** (growing/decaying)
This plasma model can have either 0, 1 or 2 wave modes.

The modes are illustrated in the CMA diagram.

3 symbols representing different types of anisotropy:
- ellipse: 0
- "eight": 8
- "infinity": \( \infty \)

When moving in the diagram, mode disappear/appear at:
- resonances; \( k \to \infty \)
- cut-offs; \( k \to 0 \)
Short summary of plasma waves

- Plasmas are ionised gases – both electromagnetic and acoustic
- Many plasma model – wide range on phenomena
- It is useful to categorise plasma waves by:
  - **Cold / warm** – how do the electrons respond to electron perturbation
    - Cold: screens E-field if perturbation is slower than $1/\omega_{pe}$
    - Warm: cannot screen short wave length $k^{-1} \lesssim \lambda_D$ (in addition to $\omega \gtrsim \omega_{pe}$)
  - **Magnetised / unmagnetised**:
    - Response becomes gyrotropic; add characteristic frequencies $\Omega_i$ and $\Omega_e$
    - Alfven wave: shear waves (like strings) & fast compressional waves
  - **High / low** $\omega$: compared to $\omega_p, \Omega_e$ and $\Omega_i$.
    - For magnetised plasmas, waves in $\Omega_i < \omega < \Omega_e, \omega_p$ are not treated here.

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<thead>
<tr>
<th></th>
<th>High freq.</th>
<th>Low freq.</th>
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<tbody>
<tr>
<td>Cold, $\mathbf{B} = 0$</td>
<td>Modified light waves</td>
<td>No waves!</td>
</tr>
<tr>
<td>Warm, $\mathbf{B} = 0$</td>
<td>Mod. light and Langmuir waves</td>
<td>Ion-acoustic waves</td>
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<tr>
<td>Cold, $\mathbf{B} \neq 0$</td>
<td>Magnetoionic &amp; cold plasma</td>
<td>Cold plasma &amp; MHD: Alfven waves</td>
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<tr>
<td>Warm, $\mathbf{B} \neq 0$</td>
<td>Not in this course</td>
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