EP2260 Lecture 4-6 2012 ©

- Utilization example

$\lambda = 4$ arrivals / hour
$\bar{x} = \frac{1}{2}$ hour
$m = 2$

offered load $a = \lambda\bar{x} = \frac{4}{3}$ Erlang

$\rho = \frac{4 \cdot T \cdot \frac{1}{3}}{T} \cdot \frac{1}{2} = \frac{2}{3}$ (< 1 !)

Back after little : $N_s = \lambda\bar{x} = \frac{4}{3}$

- Little's result $N = \lambda T$

$N(\tau) = A(\tau) - D(\tau)$

$$\int_0^t N(\tau)d\tau = \underbrace{\sum_{k=1}^{D(t)} T_k \cdot 1}_{\text{left}} + \underbrace{\sum_{k=D(t)+1}^{A(t)} (t - t_k) \cdot 1}_{\text{still in system}}$$

$$\frac{1}{t}\int_0^t N(\tau)d\tau = \frac{A(t)}{t} \underbrace{\left[\sum + \sum\right]}_{\text{sum of all system times}} \cdot \underbrace{\frac{1}{A(t)}}_{\text{all requests}}$$

$t \to \infty$

$$\boxed{N = \lambda \bullet T}$$

, $\left(\frac{1}{t}\right)$, $\frac{A(t)}{A(t)}$

Average ~ arrival intensity

- PASTA for M/M/*

$A(t, t+\Delta t)$: event of arrival in $(t, t+\Delta t)$.

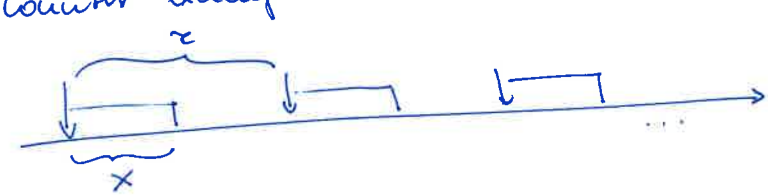
Poisson arrival (state indep) : $P[A(t,t+\Delta t) \mid N(t)=k] = P[A(t,t+\Delta t)]$
↑ state is k!

$$a_k(t) = \lim_{\Delta t \to 0} P[N(t)=k \mid A(t,t+\Delta t)] =$$

$$\lim_{\Delta t \to 0} \frac{P[N(t)=k, A(t,t+\Delta t)]}{P[A(t,t+\Delta t)]} =$$

$$\lim_{\Delta t \to 0} \frac{P[A(t,t+\Delta t) \mid N(t)=k]\, P[N(t)=k]}{P[A(t,t+\Delta t)]} = P[N(t)=k] = P_k(t)$$

- Counter example: D/D/1

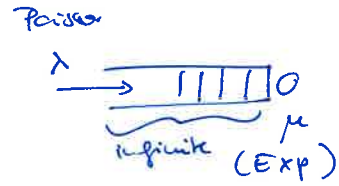


$P_0 = \frac{\tau - x}{\tau}$, $P_1 = \frac{x}{\tau}$

$a_0 = 1$, $a_1 = 0$

- M/M/1 derivation

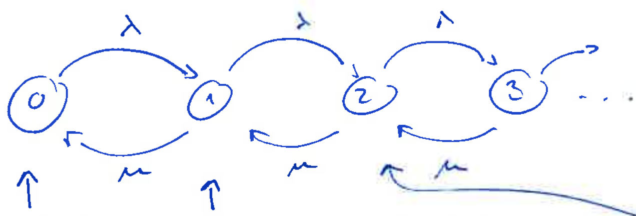
Block diagram.



Av. service time $= \bar{x} = \frac{1}{\mu}$

0. Markov chain representation

State: number of customers in the system.



Empty — 1 customer under service, noone waiting — one under service, one waiting.

Offered load: $a = \lambda \bar{x} = \frac{\lambda}{\mu}$

utilization: $\rho = \frac{a}{m} = a = \frac{\lambda}{\mu}$

1. Stationary state, state probabilities $(p = \{p_0, p_1 \dots\})$

- Condition for stability? We guess $\rho = \frac{\lambda}{\mu} < 1$

- Local balance equation:

$\lambda p_0 = \mu p_1 \Rightarrow p_1 = \frac{\lambda}{\mu} p_0 = \rho \cdot p_0$

$\lambda p_1 = \mu p_2 \qquad p_2 = \frac{\lambda}{\mu} p_1 = \rho^2 p_0$

$\vdots$

$\sum_{k=0}^{\infty} p_i = 1 \qquad p_k = \rho^k p_0 \Rightarrow p_k = (1-\rho)\rho^k$

$p_0 = \frac{1}{1+\sum_{i=1}^{\infty}\rho^i} = (1-\rho)$ [Maths sense: this is the prob. that the system is empty]

$\to p_0 = \frac{1}{1+\sum_{1}^{\infty}\rho^i} = \frac{1}{1+\frac{\rho}{1-\rho}} = \frac{(1-\rho)}{1-\rho+\rho} = 1-\rho$

Stability: to have a $\overset{\text{positive}}{}$ solution for p,

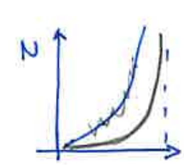
$$\boxed{\rho < 1 \Rightarrow \lambda < \mu}$$

2. Average number of customers (N)

- Note: state: # customers in the system

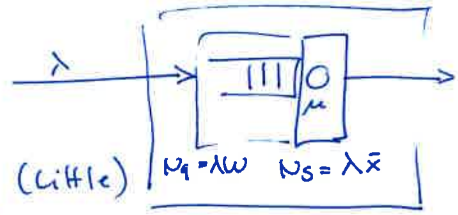
$$\underline{N} \triangleq \sum_{k=0}^{\infty} k\, p_k = \underbrace{\sum_{k=0}^{\infty} k \cdot \rho^k \cdot (1-\rho)}_{\text{we need to use sum of derivatives}} = \sum_{k=1}^{\infty} k \rho^k (1-\rho) = (1-\rho)\rho \sum_{k=1}^{\infty} \underbrace{k \cdot \rho^{k-1}}_{(\rho^k)'} =$$

$$= (1-\rho)\cdot\rho\left(\sum_{k=1}^{\infty}\rho^k\right)' = (1-\rho)\rho\left(\frac{\rho}{1-\rho}\right)' = \cancel{(1-\rho)}\rho\, \frac{\overbrace{1(1-\rho)-\rho(-1)}^{1}}{(1-\rho)^{\cancel{2}}} = \underline{\frac{\rho}{1-\rho}}$$



3. <u>Other measures</u> N_s, N_q, T, W

- Use little theorem



$$\underline{T} = \frac{N}{\lambda} = \frac{\rho}{1-\rho}\cdot\frac{1}{\lambda} = \frac{\frac{1}{\mu}}{1-\rho} = \frac{1}{\mu(1-\frac{\lambda}{\mu})} = \underline{\frac{1}{\mu-\lambda}} \quad \text{(Little)}$$

$N = \lambda \cdot T$

$W = T - \bar{x}$ $\qquad \bar{x} = \frac{1}{\mu}$

$N_s = \lambda \cdot \bar{x}$ (Little)

$N_q = \lambda W$ or $N_q = N - N_s$ or $N_q \triangleq \sum_{k=1}^{\infty} (k-1) p_k$

4. Effect or scheduling discipline? (FIFO, LIFO, random ...)

- p_k does not depend on scheduling disc.

$\Rightarrow$ average measures do not depend on it either.

(see M/M/1 ~~exercises~~ problem set)

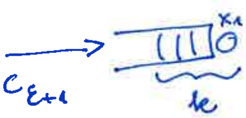
5. <u>System time distribution</u>

$T(t) = P(\text{system time} \leq t)$, $f_T(t) = \frac{dT(t)}{t}$

Laplace transform:

$Exp(\mu) \Leftrightarrow \frac{\mu}{s+\mu}$

conv. $\Leftrightarrow$ product



X_i: service time of req i

$$\underbrace{T_k = X_1 + X_2 + \dots + X_{k+1}}$$

move to laplace domain

$$\mathcal{L}(f_T(t|k)) = \prod_{k=1}^{k+1} \mathcal{L}(f(x)) = \left(\frac{\mu}{s+\mu}\right)^{k+1}$$

Probability that a new customer finds k request in the system: $a_k = p_k$ PASTA

$$\underline{\mathcal{L}(f_T(t))} = \sum_{k=0}^{\infty} \mathcal{L}(f_T(t|k))\, p_k = \sum_{k=0}^{\infty} \left(\frac{\mu}{s+\mu}\right)^{k+1} (1-\rho)\rho^k = (1-\rho)\frac{\mu}{s+\mu} \sum_{k=0}^{\infty} \left(\frac{\mu}{s+\mu}\rho\right)^k$$

$$= (1-\rho)\frac{\mu}{s+\mu}\,\frac{1}{1-\frac{\lambda}{s+\mu}} = (1-\rho)\frac{\mu}{\cancel{s+\mu}}\cdot\frac{\cancel{s+\mu}}{s+\mu-\lambda} = \underline{\frac{\mu-\lambda}{s+(\mu-\lambda)}}$$

move back to time domain:

$\boxed{T \sim Exp(\mu-\lambda)}$ $\quad (T(t) = 1 - e^{-(\mu-\lambda)t},\ f(t) = (\mu-\lambda)\, e^{-(\mu-\lambda)t}$

<u>Similarly waiting time</u>

$$W(t) = P(\text{waiting time} \leq t) = 1 - \rho\, e^{-(\mu-\lambda)t}$$