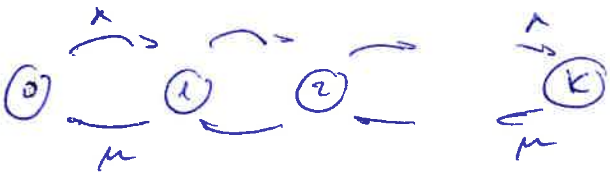
2016

M/M/1/K



$\varrho = \frac{\lambda}{\mu}$ Always stable

① $\lambda p_0 = \mu p_1$

$\vdots$

$\sum_{k=0}^{K} p_k = 1$

$$p_0 \sum_{k=0}^{\infty} \varrho^k = p_0 \left[\sum_{k=0}^{\infty} \varrho^k - \sum_{k=K+1}^{\infty} \varrho^k \right] =$$

$$p_0 \left[\frac{1}{1-\varrho} - \frac{\varrho^{K+1}}{1-\varrho} \right] \Rightarrow p_0 = \frac{1-\varrho}{1-\varrho^{K+1}}$$

② Blocking probability:

$$a_K = p_K = \frac{(1-\varrho)\varrho^K}{1-\varrho^{K+1}}$$

③ Effective traffic $\lambda_{eff} = (1-p_K)\lambda$

utilization $U = \lambda_{eff} \frac{1}{\mu}$ $[= 1 - p_0]$

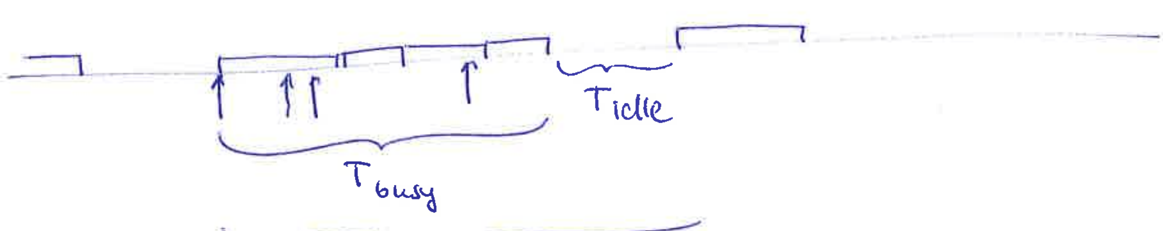
$\bar{N} \triangleq \sum k p_k$, $\bar{N}_S = \lambda_{eff} \bar{x}$ ... others with Little.

④ Length of idle, busy and blocking period:

Average time the system is idle: $\bar{T}_{idle} = \frac{1}{\lambda}$ $[Exp(\lambda)]$

Average time the system is in blocking state: $\bar{T}_{block} = \frac{1}{\mu}$ $[Exp(\mu)]$

Average time the system is busy?



This should reflect utilization in average!

$$U = \frac{\bar{T}_{busy}}{\bar{T}_{idle} + \bar{T}_{busy}} \Rightarrow \bar{T}_{busy} = \frac{U \cdot \bar{T}_{idle}}{1 - U}$$

~~(For M/M/1 $U = \varrho$, $\bar{T}_{busy} = \frac{\frac{\lambda}{\mu}\frac{1}{\lambda}}{1-\frac{\lambda}{\mu}} = \frac{1}{\mu - \lambda}$~~

2016

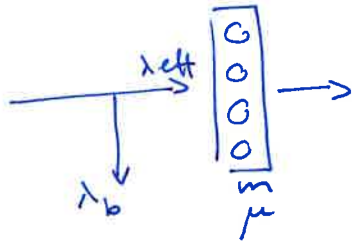
M/M/m/m : Erlang ~~block~~ loss system)

Erlang (Danish) 1878–1929

1. System definition

- Arrival: Poisson (λ)
- Service: $Exp(\mu)$, $\bar{x} = \frac{1}{\mu}$
- m servers (selected randomly)
- no buffer

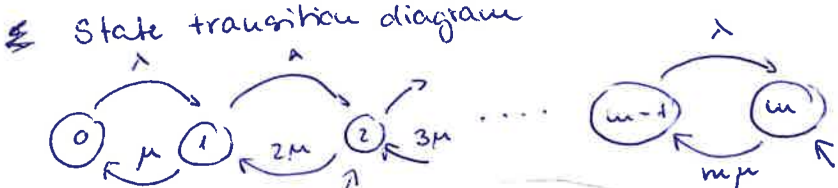
Offered load $a = \lambda \bar{x} = \frac{\lambda}{\mu}$ [Erlang]



Steady state always exist

State: # customers in the system = # occupied servers

2. State transition diagram



↖ Blocking state

Minimum of 2 $Exp(\mu)$ $\Rightarrow Exp(2\mu)$

$P(\tau < t) = P(\text{at least one service}) = 1 - P(\text{no service}) = 1 - \prod P(\text{server } i \text{ has not finished}) = 1 - \prod e^{-\mu t} = 1 - e^{-\sum \mu_i t} \Rightarrow Exp(i\mu)$

2. ~~Balance~~ Steady state prob. p

$p_0 \lambda = p_1 \mu$ $\qquad p_1 = \frac{\lambda}{\mu} p_0 = a \cdot p_0$

$p_1 \lambda = p_2 \cdot 2\mu$ $\qquad p_2 = \frac{\lambda}{2\mu} p_1 = \frac{a^2}{2} p_0$

$p_2 \lambda = p_3 \cdot 3\mu$ $\qquad p_3 = \frac{\lambda}{3\mu} p_2 = \frac{a^3}{3!} p_0$

$\vdots$

$p_m \lambda = p_m m\mu$ $\qquad p_m = \frac{a^m}{m!} p_0$

$\sum_{k=0}^{m} p_k = 1$

(group work)

$\sum_{k=0}^{m} \frac{a^k}{k!} p_0 = 1 \Rightarrow p_0 = \frac{1}{\sum_{k=0}^{m} \frac{a^k}{k!}}$

$$\boxed{p_k = \frac{a^k/k!}{\sum_{i=0}^{m} a^i/i!}}$$

Note: $m \to \infty$ $\quad p_k = \frac{a^k/k!}{e^a}$

$\Leftarrow$ truncated Poisson

3. Performance measures

$P(\text{blocking}) = P(\text{arrival}) \overset{PASTA}{=} p_m = \frac{a^m/m!}{\sum_i a^i/i!} = E_m(a) = B(m,a)$ $\quad$ $\boxed{\text{Erlang-B form}}$

$W = 0, \; N_q = 0$

$T = \bar{x} = \frac{1}{\mu}$, $\quad N = N_s \overset{Little}{=} \lambda_{eff} \bar{x} = (1-p_m)\lambda\bar{x} = (1-p_m) \cdot a$, $\quad$ utilization: $\rho = \frac{\lambda_{eff} \cdot \bar{x}}{m} = (1-p_m) \cdot \frac{a}{m}$ (<1)

4. Erlang table

$a = A = 3$, $p(\text{block}) < 0.1 \Rightarrow m = 6$

$a = A = 6$, $p(\text{block}) < 0.1 \Rightarrow m = 9$

↑ e.g. for double λ

$\Rightarrow$ efficiency of multiplexing

M/M/∞/∞