System C is an M/M/1-system with 2 queueing places. Determine the stationary probability distribution from the state diagram.

Result: $p_0 = \frac{1}{40}$, $p_1 = \frac{3}{40}$, $p_2 = \frac{9}{40}$ and $p_3 = \frac{27}{40}$

$$\bar{N}_{s,C} = 0 \cdot p_0 + 1 \cdot (p_1 + p_2 + p_3) = 0.975$$

This means that $x_{BC} = \dfrac{\bar{N}_{s,B} + \bar{N}_{s,C}}{\lambda} = 0.1175$ seconds.

The total average service time for a customer is $x_{tot} = x_A + x_{BC} \approx 0.20$ seconds.

12.

(a) $$\begin{cases} \lambda_i = \dfrac{1}{M} \cdot \left(\lambda + \alpha \cdot \sum_{i=1}^{M} \lambda_i\right) \\ \sum_{i=1}^{M} \lambda_i(1-\alpha) = \lambda \end{cases}$$ which means that $\lambda_i = \dfrac{1}{M(1-\alpha)} \cdot \lambda$ for $i=1..M$

(b) Let $\rho_i = \dfrac{\lambda_i}{\mu} = \dfrac{\lambda}{\mu M(1-\alpha)}$ for $i=1..M$.

The average number of jobs in system i is $\bar{N}_i = \dfrac{\rho_i}{1-\rho_i} = \dfrac{\lambda}{\mu M(1-\alpha)-\lambda}$ for $i=1..M$.

This means that the total average number of jobs in the network,

$$\bar{N}_{tot} = \sum_{i=1}^{M} \bar{N}_i = \frac{M\lambda}{\mu M(1-\alpha)-\lambda}$$

Let T =total average time in the network for a job.

Little's theorem gives that $T = \bar{N}_{tot}/\lambda$ which means that $T = \dfrac{M}{\mu M(1-\alpha)-\lambda}$.

(c) Assume that a job is served B times. Then, $P(B=k) = \alpha^{k-1} \cdot (1-\alpha)$.
None of these services can be in system 1.

P(a job chooses not system 1)= $1 - \dfrac{1}{M} = \dfrac{M-1}{M}$ which means that

P(a job is never served in system $1|B = k$) $= \left(\dfrac{M-1}{M}\right)^k$.

The theorem of total probability gives that P(a job is never served in system 1)=

$$\sum_{k=1}^{\infty} \left(\frac{M-1}{M}\right)^k \cdot \alpha^{k-1} \cdot (1-\alpha) = \frac{(M-1)(1-\alpha)}{M-\alpha(M-1)}$$

Solutions for the exercices in Chapter 7 (Probability Theory)

1.

(a) $P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

(b) $P(X=1|Y=0) = \frac{P(X=1, Y=0)}{P(Y=0)} = \frac{P(X=1, Y=0)}{P(Y=0, X=0) + P(Y=0, X=1)} = \ldots = \frac{1}{5}$

(c) $P(Y=0|X=1) = \frac{P(Y=0, X=1)}{P(X=1)} = \frac{P(Y=0, X=1)}{1-P(X=0)} = \frac{1}{2}$

(d) $E[Y] = 0 \cdot P(Y=0) + 1 \cdot P(Y=1) = P(Y=1) = \ldots = \frac{3}{8}$

(e) $E[XY] = 0 \cdot P(XY=0) + 1 \cdot P(XY=1) = P(XY=1) = P(X=1, Y=1) = \frac{1}{8}$

(f) $E[X|Y=0] = 0 \cdot P(X=0|Y=0) + 1 \cdot P(X=1|Y=0) = \ldots = \frac{1}{5}$

(g) $E[Y|X=1] = 0 \cdot P(Y=0|X=1) + 1 \cdot P(Y=1|X=1) = 1 - P(Y=0|X=1) = \frac{1}{2}$

2. Assume a random variable X that is geometrically distributed with parameter p.

$$E[X] = \sum_{k=1}^{\infty} k \cdot P(X=k) = p(1-p)\sum_{k=1}^{\infty} kp^{k-1}$$

The sum is one of the "predefined sums" (see the book), which means that:

$$E[X] = p(1-p)\sum_{k=1}^{\infty} kp^{k-1} = p(1-p) \cdot \frac{1}{(1-p)^2} = \frac{p}{1-p}$$

The variance of X is given by: $V[X] = E[X^2] - E[X]^2$, where

$$E[X^2] = \sum_{k=1}^{\infty} k^2 \cdot P(X=k) = \sum_{k=1}^{\infty} k^2 p^k (1-p) = p(1-p)\sum_{k=1}^{\infty} k^2 p^{k-1}$$

This sum is not one of the predefined sums, however it can be identified as:

$$\sum_{k=1}^{\infty} k^2 p^{k-1} = \frac{\partial}{\partial p}\left(\sum_{k=1}^{\infty} kp^k\right) = \frac{\partial}{\partial p}\left(p\sum_{k=1}^{\infty} kp^{k-1}\right) = \frac{\partial}{\partial p}\left(\frac{p}{(1-p)^2}\right) = \frac{(1-p)^2 + p \cdot 2(1-p)}{(1-p)^4}$$

This means that: $E[X^2] = p(1-p) \cdot \frac{(1-p)^2 + p \cdot 2(1-p)}{(1-p)^4} = \ldots = \frac{p+p^2}{(1-p)^2}$

and that $V[X] = \frac{p+p^2}{(1-p)^2} - \left(\frac{p}{1-p}\right)^2 = \ldots = \frac{p}{(1-p)^2}$

3. The squared coefficient of variance is given by $C^2 = \dfrac{V[X]}{E[X]^2}$

(a) The density function for a exponentially distributed random variable, X, with mean $1/\lambda$ is given by $f(x) = \lambda \cdot e^{-\lambda x}$, and the Laplace transform is $F^*(s) = \dfrac{\lambda}{\lambda + s}$.

This means that $E[X^2] = F^{*\prime\prime}(0) = \left.\dfrac{2\lambda}{(\lambda + s)^3}\right|_{s=0} = \dfrac{2}{\lambda^2}$ and that

$$C^2 = \frac{V[X]}{E[X]^2} = \frac{E[X^2] - E[X]^2}{E[X]^2} = \frac{1/\lambda^2}{1/\lambda^2} = 1$$

(b) The Laplace transform for an Erlangian-r distributed random variable, X, with mean $1/\lambda$ is given by $F^*(s) = \left(\dfrac{r\lambda}{r\lambda + s}\right)^r$.

This means that $E[X^2] = F^{*\prime\prime}(0) = \left.\dfrac{r(r+1)(r\lambda)^r}{(r\lambda + s)^{-(r+2)}}\right|_{s=0} = \dfrac{1}{\lambda^2} + \dfrac{1}{r\lambda^2}$ and that

$$C^2 = \frac{V[X]}{E[X]^2} = \frac{\dfrac{1}{\lambda^2} + \dfrac{1}{r\lambda^2} - \dfrac{1}{\lambda^2}}{\dfrac{1}{\lambda^2}} = \frac{1}{r}.$$

(c) The limits for r are $1 \le r < \infty$, which means that the lower limit for C^2 is 0 and the upper limit is 1.

4. x_n is a random variable representing the number of throws until "head" comes up the n:th time.
P(head)=p, P(not head)=1-p=q
Also, define the following random variables:
y_1= number of throws until "head" for the 1st time.
y_2= number of throws between 1st and 2nd time.
y_i= number of throws between (i-1):th and i:th time.

This means that $x_n = \sum_{i=1}^{n} y_i$ and that $E[x_n] = E\left[\sum_{i=1}^{n} y_i\right]$.

Since all y_i are independent with the same probability distribution we get:

$$E\left[\sum_{i=1}^{n} y_i\right] = \sum_{i=1}^{n} E[y_i] = n \cdot E[y_1]$$

The probability distribution for y_1 is given by: $P(y_1 = k) = q^{k-1} \cdot p$, which means that:

$$E[y_1] = \sum_{k=1}^{\infty} k \cdot P(y_1 = k) = \sum_{k=1}^{\infty} k \cdot q^{k-1} \cdot p = p \cdot \sum_{k=1}^{\infty} k \cdot q^{k-1} = p \cdot \frac{1}{(1-q)^2} = \frac{1}{p}$$

This means that $E[x_n] = n \cdot E[y_1] = \dfrac{n}{p}$.

5. The probability distribution for V is defined as $P(V=k)$ for $k \geq 0$.
It is determined by using the theorem of total probability:

$$P(V=k) = \int_0^{\infty} P(V=k|X=x) \cdot f_X(x)dx$$

where $f_X(x)$ is the density function for X.

The distribution function for X is given by $F_X(x) = P(X \leq x) = 1 - e^{-\mu x}$, which means that the density function is given by $f_X(x) = \frac{d}{dx}F_X(x) = \mu e^{-\mu x}$.

This means that

$$P(V=k) = \int_0^{\infty} \frac{(\lambda x)^k}{k!} \cdot e^{-\lambda x} \cdot \mu e^{-\mu x} dx = \frac{\lambda^k}{k!}\mu \int_0^{\infty} x^k \cdot e^{-(\lambda+\mu)x} dx$$

Solve the integral first:

$$\int_0^{\infty} x^k \cdot e^{-(\lambda+\mu)x} dx = \left[-\frac{1}{(\lambda+\mu)} \cdot e^{-(\lambda+\mu)x} \cdot x^k\right]_0^{\infty} + \int_0^{\infty} kx^{k-1} \cdot \frac{1}{(\lambda+\mu)} e^{-(\lambda+\mu)x} dx$$

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$$= \ldots = \int_0^{\infty} \frac{k!}{(\lambda+\mu)^k} \cdot e^{-(\lambda+\mu)x} dx = \frac{k!}{(\lambda+\mu)^k}\left[-\frac{1}{(\lambda+\mu)} \cdot e^{-(\lambda+\mu)x}\right]_0^{\infty}$$

$$= \frac{k!}{(\lambda+\mu)^k} \cdot \frac{1}{\lambda+\mu}$$

This means that $P(V=k) = \frac{\lambda^k}{k!}\mu \cdot \frac{k!}{(\lambda+\mu)^k} \cdot \frac{1}{\lambda+\mu} = \left(\frac{\lambda}{\lambda+\mu}\right)^k \cdot \frac{\mu}{\lambda+\mu}$.

6. The probability distribution for N is given by: $p_k = P(N=k) = \frac{m^k}{k!} \cdot e^{-m}$.

The z-transform for N is given by:

$$P(z) = \sum_{k=0}^{\infty} p_k \cdot z^k = \sum_{k=0}^{\infty} \frac{m^k}{k!} \cdot e^{-m} \cdot z^k = e^{-m} \cdot \sum_{k=0}^{\infty} \frac{(mz)^k}{k!} = e^{-m} \cdot e^{mz} = e^{m(z-1)}$$

The mean of N is determined by: $E[N] = \lim_{z \to 1} \frac{d}{dz}P(z) = \lim_{z \to 1} m \cdot e^{m(z-1)} = m$

The variance of N is given by $V[N] = E[N^2] - E[N]^2$

where $E[N^2] = \lim_{z \to 1} \frac{d^2}{dz^2}P(z) + E[N] = \lim_{z \to 1} m^2 \cdot e^{m(z-1)} + m = m^2 + m$,

which means that $V[N] = m^2 + m - m^2 = m$.

7. The probability distributions for X and Y are given by:

$$P(X=k) = \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_1} \qquad P(Y=k) = \frac{\lambda_2^k}{k!} \cdot e^{-\lambda_2}$$

Since $Z=X+Y$, the z-transform for Z, $P_Z(z)$, is given by $P_Z(z) = P_X(z) \cdot P_Y(z)$, where $P_X(z)$ and $P_Y(z)$ are the z-transforms for X and Y respectively.

These are given by: $P_X(z) = e^{\lambda_1(z-1)}$ $\qquad P_Y(z) = e^{\lambda_2(z-1)}$.

which means that $P_Z(z) = e^{\lambda_1(z-1)} \cdot e^{\lambda_2(z-1)} = e^{(\lambda_1+\lambda_2)(z-1)}$.

The probability distribution is found by inverse transform of $P_Z(z)$ and the result is:

$$P(Z = k) = \frac{(\lambda_1 + \lambda_2)^k}{k!} \cdot e^{-(\lambda_1+\lambda_2)}$$

This means that Z is Poissonian distributed with mean $\lambda_1 + \lambda_2$.