which means that $P_Z(z) = e^{\lambda_1(z-1)} \cdot e^{\lambda_2(z-1)} = e^{(\lambda_1+\lambda_2)(z-1)}$.

The probability distribution is found by inverse transform of $P_Z(z)$ and the result is:

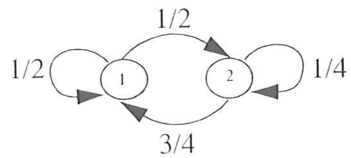
$$P(Z = k) = \frac{(\lambda_1 + \lambda_2)^k}{k!} \cdot e^{-(\lambda_1 + \lambda_2)}$$

This means that Z is Poissonian distributed with mean $\lambda_1 + \lambda_2$.

Solutions for the exercices in Chapter 8 (Stochastic processes)

1. The system has two states, 1 and 2

   (a) The state diagram is:



   (b) The stationary probability distribution is given by $\mathbf{p} \cdot \mathbf{P} = \mathbf{P}$ which means that

$$\begin{cases} p_1 = \frac{1}{2}p_1 + \frac{3}{4}p_2 \\ p_2 = \frac{1}{2}p_1 + \frac{1}{4}p_2 \end{cases} \Rightarrow p_1 = \frac{3}{2}p_2$$

   Two find the exact value of p_j, the normalisation condition must be used:

$$\sum_{k=1}^{2} p_k = 1 \Rightarrow p_1 + p_2 = 1 \Rightarrow \frac{3}{2}p_2 + p_2 = 1 \Rightarrow \begin{cases} p_1 = \frac{3}{5} \\ p_2 = \frac{2}{5} \end{cases}$$

   (c) For a discrete-time Markov chain the transient probability distribution is given by:

$$\mathbf{p}(n) = \mathbf{p}(0) \cdot \mathbf{P}^n$$

   In this case, $\mathbf{p}(0)=(1,0)$, which means that:

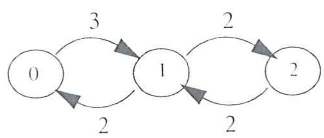
$$\mathbf{p}(1) = \mathbf{p}(0) \cdot \mathbf{P} \Rightarrow \mathbf{p}(1) = (1,0) \cdot \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = (\frac{1}{2},\frac{1}{2})$$

$$\mathbf{p}(2) = \mathbf{p}(1) \cdot \mathbf{P} \Rightarrow \mathbf{p}(2) = (\frac{1}{2},\frac{1}{2}) \cdot \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = (\frac{5}{8},\frac{3}{8})$$

$$\mathbf{p}(3) = \mathbf{p}(2) \cdot \mathbf{P} \Rightarrow \mathbf{p}(2) = (\frac{5}{8},\frac{3}{8}) \cdot \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = (\frac{19}{32},\frac{13}{32})$$

2. The system has 3 states: 0, 1, and 2.

   (a) The state diagram is:



(b) The stationary probability distribution, $\mathbf{p}=(p_0, p_1, p_2)$, is calculated as:

$$\mathbf{p}Q = 0 \Rightarrow \begin{cases} -3p_0 + 2p_1 & = 0 \\ 3p_0 - 4p_1 + 2p_2 & = 0 \\ 2p_1 - 2p_2 & = 0 \end{cases} \Rightarrow \begin{cases} p_1 = \frac{3}{2}p_0 \\ p_2 = \frac{3}{2}p_0 \end{cases}$$

To find p_0 the normalisation condition must be used:

$$\sum_{k=0}^{2} p_k = 1 \Rightarrow p_0 + \frac{3}{2}p_0 + \frac{3}{2}p_0 = 1 \Rightarrow \begin{cases} p_0 = 1/4 \\ p_1 = 3/8 \\ p_2 = 3/8 \end{cases}$$

(c) The z-transform is defined as: $P(z) = \sum_{k=0}^{2} p_k \cdot z^k = \frac{1}{4} + \frac{3}{8}z + \frac{3}{8}z^2$.

(d) Let N be a random variable for the number of jobs in the system. The stationary probability distribution of N is given by p_k above.
Alternative 1:

$$E[N] = \sum_{k=0}^{2} k \cdot p_k = p_1 + 2p_2 = \frac{9}{8}$$

$$E[N^2] = \sum_{k=0}^{2} k^2 \cdot p_k = p_1 + 4p_2 = \frac{15}{8}$$

$$V[N] = E[N^2] - E[N]^2 = \frac{15}{8} - \frac{81}{64} = \frac{39}{64}$$

Alternative 2:

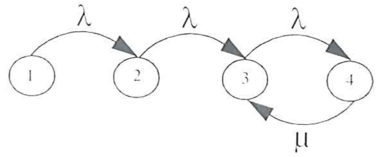
$$E[N] = \lim_{z \to 1} \frac{d}{dz}P(z) = \lim_{z \to 1}\left(\frac{3}{8} + \frac{6}{8}z\right) = \frac{9}{8}$$

$$E[N^2] = \lim_{z \to 1} \frac{d^2}{dz^2}P(z) + E[N] = \lim_{z \to 1}\left(\frac{6}{8}\right) + \frac{9}{8} = \frac{15}{8}$$

$$V[N] = E[N^2] - E[N]^2 = \frac{15}{8} - \frac{81}{64} = \frac{39}{64}$$

3. The system has four states: 1..4.

(a) The state diagram for the system:



(b) The transient probability distribution, $\mathbf{p}(t)=[p_1(t), p_2(t), p_3(t), p_4(t)]$, is determined by the following equation:

$$\frac{d}{dt}\mathbf{p}(t) = \mathbf{p}(t) \cdot Q$$

The equations for $p_1(t)$ and $p_2(t)$ are:

$$\begin{cases} \frac{d}{dt}p_1(t) = -\lambda p_1(t) \\ \frac{d}{dt}p_2(t) = \lambda p_1(t) - \lambda p_2(t) \end{cases}$$

Assume that $\mathbf{p}(0)=[1, 0, 0, 0]$, which means that we start in state 1. Laplace transform of the differential equations give::

$$\begin{cases} sP_1(s) - 1 = -\lambda P_1(s) \\ sP_2(s) - 0 = \lambda P_1(s) - \lambda P_2(s) \end{cases}$$

After solving the equations and inversing the Laplace transforms we get:

$$\begin{cases} p_1(t) = e^{-\lambda t} \\ p_2(t) = \lambda t e^{-\lambda t} \end{cases}$$

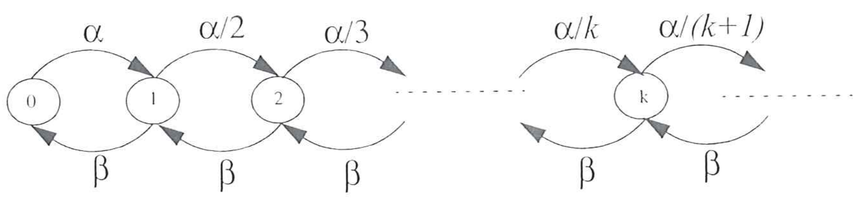
(c) The stationary probability distribution, $\mathbf{p}=[p_1, p_2, p_3, p_4]$, is determined by solving $\mathbf{p} \cdot Q = 0$.
Since the states 1 and 2 are so-called transient states is $p_1=p_2=0$. p_3 and p_4 are determined by solving the following equations::

$$\begin{cases} p_4 = \frac{\lambda}{\mu} \cdot p_3 \\ p_3 + p_4 = 1 \end{cases}$$

This means that $p_3 = \frac{\mu}{\lambda + \mu}$ and $p_4 = \frac{\lambda}{\lambda + \mu}$.

4. The system has an infinite number of states, which means that the Q-matrix becomes impractical.

(a) State diagram for the system:



(b) By using the so-called "cut-method" the following balance equations are derived:

$$\begin{cases} \alpha p_0 = \beta p_1 \\ \frac{\alpha}{2} \cdot p_1 = \beta p_2 \\ \vdots \\ \frac{\alpha}{k} \cdot p_{k-1} = \beta p_k \end{cases}$$

By using substitution, the following general formula can be found:

$$p_k = \left(\frac{\alpha}{\beta}\right)^k \cdot \frac{1}{k!} \cdot p_0 \qquad k \geq 0$$

p_0 is determined by using the normalisation condition:

$$\sum_{k=0}^{\infty} p_k = 1 \Rightarrow \sum_{k=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^k \cdot \frac{1}{k!} \cdot p_0 = 1 \Rightarrow p_0 = \frac{1}{\sum_{k=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^k \cdot \frac{1}{k!}} = e^{-\frac{\alpha}{\beta}}$$

(c) Let N be a random variable that represent the number of jobs in the system. Then,

$$E[N] = \sum_k k \cdot P(N = k) = \sum_{k=0}^{\infty} k \cdot p_k = \frac{\alpha}{\beta} \cdot e^{-\frac{\alpha}{\beta}} \cdot \sum_{k=1}^{\infty} \left(\frac{\alpha}{\beta}\right)^{k-1} \cdot \frac{1}{(k-1)!} = \frac{\alpha}{\beta}$$

5.

(a) Stationary probability distribution: p_1=0, p_2=0, p_3=0, p_4=1 since the system cannot move from state 4.

(b) The total transition intensity is λ+μ.

(c) Let T be a random variable for the time spent in state 3.
Let X_4 be the time until the system will change to state 4 and
X_1 be the time until the system will change to state 1. X_4 and X_1 are independent random variables that are exponentially distributed with means $1/\lambda$ and $1/\mu$ respectively (since we have a Markov chain).
Then T=min(X_1, X_4), which means that:

$$P(T \leq t) = P(min(X_1, X_2) \leq t) = 1 - P(min(X_1, X_2) > t) = 1 - P(X_1 > t)P(X_4 > t)$$
$$= 1 - e^{-\mu t}e^{-\lambda t} = 1 - e^{-(\lambda + \mu)t}$$

which means that T is exponentially distributed with mean $1/(\lambda + \mu)$.

(d) Let T_k=average time in state k. Then:

$$T_1 = \frac{1}{2\lambda} \qquad T_2 = \frac{1}{\lambda} \qquad T_3 = \frac{1}{\lambda + \mu} \qquad T_4 = \infty$$

(e) P(transition from state 3 to state 4)=$\frac{\lambda}{\lambda + \mu}$

P(transition from state 3 to state 1)=$\frac{\mu}{\lambda + \mu}$

(f) Lett M_k=the average time to reach state 4 if we start in state k. This means that we can derive the following equation system:

$$\begin{cases} M_1 = T_1 + M_2 \\ M_2 = T_2 + M_3 \\ M_3 = T_3 + \dfrac{\lambda}{\lambda+\mu} \cdot M_4 + \dfrac{\mu}{\lambda+\mu} \cdot M_1 \\ M_4 = 0 \end{cases}$$

Solve the equations to determine M_1. Result: $M_1 = \dfrac{5\lambda + 3\mu}{2\lambda^2}$

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① upload the formula sheet.

② upload the Lecture Notes.