Quarter wave plates and Jones calculus for optical system

T. Johnson

Outline

- Quarter wave plates – map circular to linear polarisation!
- **Jones calculus**; matrix formulation of how wave polarization changes when passing through polarizing component
  - Examples: linear polarizer, quarter wave plate

Next lecture:

- Statistical representation of incoherent/unpolarized waves
- Stokes vector and polarization tensor
  - Poincare sphere
- **Muller calculus**;
  matrix formulation
  for the transmission of
  partially polarized waves

Birefrigent media

- Previous lecture we noted that in birefringent crystals:
  - Orientate optical axis in the z-direction

$$K(\omega) = \begin{pmatrix} K_\perp(\omega) & 0 & 0 \\ 0 & K_\perp(\omega) & 0 \\ 0 & 0 & K_\parallel(\omega) \end{pmatrix}$$

  - Orient $\mathbf{k}$ perpendicular to the y-axis: $\kappa = (\sin\theta \; , 0 \; , \; \cos\theta)$
  - The two modes, O-mode and X-mode, are described by:

$$\begin{cases} n_O{}^2 = K_\perp \\ n_X{}^2 = \dfrac{K_\perp K_\parallel}{K_\perp \sin^2\theta + K_\parallel \cos^2\theta} \end{cases} \qquad \begin{cases} \mathbf{e}_O(\mathbf{k}) = (0 \; , 1 \; , 0) \\ \mathbf{e}_X(\mathbf{k}) \propto (K_\parallel \cos\theta \; , \; 0 \; , K_\perp \sin\theta) \end{cases}$$

  - thus if $K_\perp > K_\parallel$ then $n_O \geq n_X$
    the O-mode has larger phase velocity

The quarter wave plate

- Important use of birefringent media: Quarter wave plates
  - uniaxial crystal; normal in z-direction
  - length/width L in the x-direction:

$$L = \frac{c}{\omega}\frac{\pi/2}{\sqrt{K_{\parallel}} - \sqrt{K_{\perp}}}$$

(why this formula is explained later!)

  - Let the waves travel in the x-direction, i.e. k is in the x-direction and $\theta=\pi/2$

$$\begin{cases} n_O{}^2 = K_{\perp} \\ n_X{}^2 = K_{\parallel} \end{cases}$$

$$\begin{cases} \mathbf{e}_O(\mathbf{k}) = (0\ ,\ 1\ ,\ 0) \\ \mathbf{e}_X(\mathbf{k}) = (0\ ,\ 0\ ,\ 1) \end{cases}$$

Exercise: Modifying polarization in a quarter wave plate (1)

Consider a uniaxial plate with axis in the z-direction. The plate is not spatially dispersive.

Let light pass through the plate with the wave vector perpendicular to the plate. Also let the incoming light be linearly polarised with a 45^o angel between the electric field and the optical axis of the plate.

Use dispersion relations and eigenvectors:

$$\begin{cases} n_O{}^2 = K_\perp \\ n_X{}^2 = K_\parallel \end{cases} \qquad \begin{cases} \mathbf{e}_O(\mathbf{k}) = (0\ ,1\ ,0) \\ \mathbf{e}_X(\mathbf{k}) = (0\ ,\ 0\ ,1) \end{cases}$$

a) Express the incoming light as a superposition of the eigenvectors.

b) Describe how the polarisation changes as the wave travels through the plate.

c) For which length of the plate is the outcoming wave circularly polarised?

NOTE: Plates that transforms linear polarisation into circular (ellipical) polarisation are called *quarter wave plates*.

What happens if the incoming light is circularly polarised?

Exercise: Modifying polarization in a quarter wave plate (2)

- Plane wave ansats has to match dispersion relation
  - when the wave enters the crystal it will slow down, this corresponds to a change in wave length, or $\mathbf{k}$

$$k_O = \frac{\omega n_O}{c} = \frac{\omega}{c}\sqrt{K_\perp} \ , \ k_X = \frac{\omega n_X}{c} = \frac{\omega}{c}\sqrt{K_\parallel}$$

  - since the O- and X-mode travel at different speeds we write

$$\mathbf{E}(t,x) = \Re\left\{\mathbf{e}_O E_O \exp(ik_O x - i\omega t) + \mathbf{e}_X E_X \exp(ik_X x - i\omega t)\right\}$$

- Assume: a linearly polarized wave enters the crystal

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} = \left(\mathbf{e}_O + \mathbf{e}_X\right) \Rightarrow E_O = E_X = 1$$

$$\mathbf{E}(t,x) = \Re\left\{\mathbf{e}_O \exp(ik_O x - i\omega t) + \mathbf{e}_X \exp(ik_X x - i\omega t)\right\}$$

$$= \mathbf{e}_O \cos(k_O x - \omega t) + \mathbf{e}_X \cos(k_O x - \omega t + \Delta k x) \ , \ \Delta k \equiv k_X - k_O$$

  - the difference in wave number causes the O- and X-mode to drift in and out of phase with each other!

Ex: Modifying wave polarization in a quarter wave plate (3)

- The polarization when the wave exits the crystal at $x=L$

$$\mathbf{E}(t,x) = \left[\mathbf{e}_O \cos(k_O L - \omega t) + \mathbf{e}_X \cos(k_O L - \omega t + \Delta k L)\right]$$

- Select plate width: $L = \frac{c}{\omega}\frac{\pi/2}{\sqrt{K_\parallel} - \sqrt{K_\perp}} \Rightarrow \Delta k L = \pi/2$

$$\Rightarrow \mathbf{E}(t,x) = \left[\mathbf{e}_O \cos(k_O L - \omega t) - \mathbf{e}_X \sin(k_O L - \omega t)\right]$$ *Circular polarisation!*

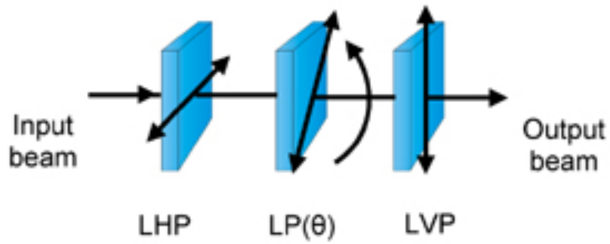
- The crystal converts the linear polarisation, $(1,1,0)$ into circular polarization
  - Cyclic mapping: $(1,1,0) \rightarrow (1,i,0) \rightarrow (1,-1,0) \rightarrow (1,-i,0) \rightarrow (1,1,0)$

    linear → *right hand circ* → *rotated linear* → *left hand circ* → *linear*
  - But $(1,0,0)$ and $(0,1,0)$ are ***unchanged*** – why?
- Called a quarter wave plate; a common component in optical systems
- But work *only* at one wave length – adapted for e.g. a specific laser!
- In general, waves propagating in birefringent crystal change polarization back and forth between linear to circular polarization
- Switchable wave plates can be made from liquid crystal
  - angle of polarization can be switched by electric control system

- Similar effect: Faraday effect in magnetoactive media (home assignment)

Optical systems

- In optics, interferometry, polarimetry, etc, there is an interest in following how the wave polarization changes when passing through e.g. an optical system.



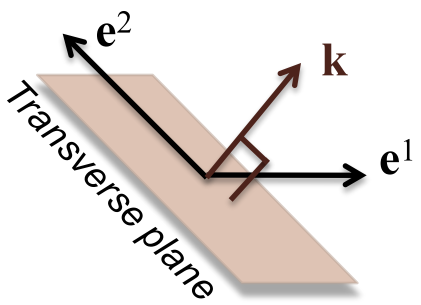
- For this purpose two types of calculus have been developed;
  - Jones calculus; only for coherent (polarized) wave
  - Muller calculus; for both coherent, unpolarised and partially polarised
- In both cases the wave is given by vectors $\mathbf{E}$ and $\mathbf{S}$ (defined later) and polarizing elements are given by matrices J and M

$$\mathbf{E}_{out} = J \bullet \mathbf{E}_{in}$$

$$\mathbf{S}_{out} = M \bullet \mathbf{S}_{in}$$

The polarization of transverse waves

- Lets first introduce a new coordinate system representing vectors in the *transverse plane*, i.e. perpendicular to the $\mathbf{k}$.
  - Construct an orthonormal basis for $\{\mathbf{e}^1,\mathbf{e}^2,\boldsymbol{\kappa}\}$, where $\boldsymbol{\kappa}=\mathbf{k}/|\mathbf{k}|$
  - The transverse plane is then given by $\{\mathbf{e}^1,\mathbf{e}^2\}$, where:



$$\mathbf{e}^\alpha = e_i^\alpha \mathbf{e}_i$$

  - where $\alpha=1,2$ and $\mathbf{e}_i$, $i=1,2,3$ is any basis for $\mathcal{R}^3$
  - denote $\mathbf{e}^1$ the *horizontal* and $\mathbf{e}^2$ the *vertical* directions
- The electric field then has different component representations: E_i (for $i=1,2,3$) and E^α (for $\alpha=1,2$)

$$E_i = e_i^\alpha E^\alpha$$

  - similar for the polarization vector, e_M

$$e_{M,i} = e_i^\alpha e_M^\alpha$$

The new coordinates provide 2D representations

Some simple Jones Matrixes

- In the new coordinate system the Jones matrix is 2x2:
  - The electric field entering an optical component is: E_{in}^{β}
  - The exiting the component is then: $E_{out}^{\alpha} = J^{\alpha\beta} E_{in}^{\beta}$
  - Here $J^{\alpha\beta}$ is the Jones matrix: $J^{\alpha\beta} = \begin{bmatrix} J^{11} & J^{12} \\ J^{21} & J^{22} \end{bmatrix}$
- Example: **Linear polarizer** transmitting Horizontal polarization, *(L,H)*

$$J^{\alpha\beta}{}_{L,H} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E^H \\ E^V \end{bmatrix} = \begin{bmatrix} E^H \\ 0 \end{bmatrix}$$

- Example: **Attenuator** transmitting a fraction ρ of the energy
  - Note: $energy \sim \varepsilon_0 |\mathbf{E}|^2$

$$J^{\alpha\beta}{}_{Att}(\rho) = \sqrt{\rho} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \sqrt{\rho} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E^H \\ E^V \end{bmatrix} = \sqrt{\rho} \begin{bmatrix} E^H \\ E^V \end{bmatrix}$$

Jones matrix for a quarter wave plate

- Quarter wave plates are birefringent (have two different refractive index)
  - align the plate such that horizontal / veritical polarization (corresponding to O/X-mode) has wave numbers $\mathbf{k}^1$ / $\mathbf{k}^2$

$$\begin{bmatrix} E^H(x) \\ E^V(x) \end{bmatrix} = \begin{bmatrix} E^H(0)\exp(ik^1x) \\ E^V(0)\exp(ik^2x) \end{bmatrix}$$

  - let the light enter the plate start at $x=0$ and exit at $x=L$

$$\mathbf{E}(L) = \begin{bmatrix} e^{ik^1L} & 0 \\ 0 & e^{ik^2L} \end{bmatrix} \begin{bmatrix} E^H(0) \\ E^V(0) \end{bmatrix} \equiv J_{Ph}\mathbf{E}(0)$$

  - where Ph stands for *phaser*
- Quarter wave plates change the relative phase by $\pi/2$

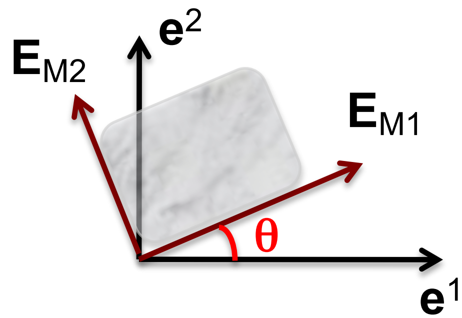
$$k^1L - k^2L = \pm\pi/2 \rightarrow J_Q = e^{ik^1L} \begin{bmatrix} 1 & 0 \\ 0 & \pm i \end{bmatrix}$$

  - usually we considers only relative phase and skip factor $\exp(ik^1L)$

Jones matrix for a rotated birefringent media

- If a birefringent media (e.g. quarter wave plates) is not aligned with the axis of our coordinate system…
  - …then we may use a rotation matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \rightarrow R^{-1}(\theta) = R(-\theta)$$



  - In the rotated system the plate is a phaser!

$$\begin{bmatrix} E_{M1}(x) \\ E_{M2}(x) \end{bmatrix} = \begin{bmatrix} e^{ik^1 x} & 0 \\ 0 & e^{ik^2 x} \end{bmatrix} \begin{bmatrix} E^1(0) \\ E^2(0) \end{bmatrix}$$

  - Relation between original and rotated systems

$$\begin{bmatrix} E^1(x) \\ E^2(x) \end{bmatrix} = R(-\theta) \begin{bmatrix} E_{M1}(x) \\ E_{M2}(x) \end{bmatrix} = R(-\theta) \begin{bmatrix} e^{ik^1 x} & 0 \\ 0 & e^{ik^2 x} \end{bmatrix} \begin{bmatrix} E^1(0) \\ E^2(0) \end{bmatrix}$$

$$= R(-\theta) \begin{bmatrix} e^{ik^1 x} & 0 \\ 0 & e^{ik^2 x} \end{bmatrix} R(\theta) \begin{bmatrix} E^1(0) \\ E^2(0) \end{bmatrix}$$

$$J_{Ph}^{\alpha\beta} = R(-\theta) \begin{bmatrix} e^{ik^1 x} & 0 \\ 0 & e^{ik^2 x} \end{bmatrix} R(\theta)$$

Summary

- Quarter wave plates splits incoming waves in O- and X-mode and…
  - Transforms circular polarisation into linear polarisation
  - Transforms linear polarisation into elliptic polarisation
  - Circular map: $(1,1,0) \to (1,i,0) \to (1,-1,0) \to (1,-i,0) \to (1,1,0)$

    linear — *right hand circ* — *rotated linear* — *left hand circ* — *linear*
- Transverse waves have $\mathbf{E}$; thus use $\mathbf{E}$ -components in the transverse plane : $\mathbf{E} = E^1\mathbf{e}^1 + E^2\mathbf{e}^2 = E^\alpha\mathbf{e}^\alpha$
  - Orientation of coordinates: $\mathbf{e}^1 \times \mathbf{e}^2 = \hat{\mathbf{k}}$; $\mathbf{e}^1$ - horizontal, $\mathbf{e}^2$ - vertical
- Optical components can be described by Jones Matrix:
  - Linear polariser

$$J_{L,H}^{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

  - Phaser (birefringent media)

$$J_{Ph}^{\alpha\beta}(kx) = \begin{bmatrix} 1 & 0 \\ 0 & \exp(ikx) \end{bmatrix}$$

  - Quarter wave plates

$$J_{Q\pm}^{\alpha\beta} = J_{Ph}^{\alpha\beta}(\pm\pi/2) = \begin{bmatrix} 1 & 0 \\ 0 & \pm i \end{bmatrix}$$

Verify that $J_{Q\pm}^{\alpha\beta}$ reproduces the circular mapping above!