Polarized and unpolarised transverse waves, with applications to optical systems

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Outline

Previous lecture:

- The *quarter wave plate*
- Set up coordinate system suitable for transverse waves
- **Jones calculus**; matrix formulation of how wave polarization changes when passing through polarizing component
  - Examples: linear polarizer, quarter wave plate, Faraday rotation

This lecture

- Statistical representation of incoherent/unpolarized waves
  - Polarization tensors and Stokes vectors
    - The Poincare sphere
- **Muller calculus**; matrix formulation for the transmission of partially polarized waves

Incoherent/unpolarised

- Many sources of electromagnetic radiation are not coherent
  - they do *not* radiate perfect harmonic oscillations (not sinusoidal wave)
    - over short time scales the oscillations look harmonic
    - but over longer periods the wave look incoherent, or even stochastic
  - such waves are often referred to as unpolarised

- To model such waves we will consider the electric field to be a stochastic process, i.e. it has
  - an average: $< E^{\alpha}(t,\mathbf{x}) >$
  - a variance: $< E^{\alpha}(t,\mathbf{x})\,[E^{\beta}(t,\mathbf{x})\,]^{*} >$
  - a covariance: $< E^{\alpha}(t,\mathbf{x})\,[E^{\beta}(t+s,\mathbf{x}+\mathbf{y})\,]^{*} >$

- In this chapter we will focus on the variance, here called the intensity tensor

$$I^{\alpha\beta} = < E^{\alpha}\,(t,\mathbf{x})\,[E^{\beta}\,(t,\mathbf{x})\,]^{*} >$$

and the polarization tensor (where $\mathbf{e}_M = \mathbf{E}\,/\,|\mathbf{E}|$ is the polarization vector)

$$p^{\alpha\beta} = < e_M{}^{\alpha}\,(t,\mathbf{x})\,[\,e_M{}^{\beta}\,(t,\mathbf{x})\,]^{*} >$$

Representations for the polarization tensor

- The polarisation tensor: $p^{\alpha\beta} = \langle e^{\alpha *} e^{\beta} \rangle$
  - It has four complex components
  - But $p^{\alpha\beta}$ is constructed from a normalised vector
  - It's components are not all independent!
  - What are the possible forms for $p^{\alpha\beta}$?
- Some restrictions/properties:
  - The tensor has unit trace: $\mathrm{tr}(\mathbf{p}) = p^{\alpha\alpha} = \langle e^{\alpha *} e^{\alpha} \rangle = \langle |\mathbf{e}| \rangle = \langle 1 \rangle = 1$
  - It's hermitian: $\left(p^{\alpha\beta}\right)^* = \langle e^{\alpha} e^{\beta *} \rangle = \langle e^{\beta *} e^{\alpha} \rangle = p^{\beta\alpha}$

- The polarisation tensor is described by three real parameter $\{q, u, v\}$

$$p^{\alpha\beta} = \frac{1}{2}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + q\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + u\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + v\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\right)$$

  – Note: the three last ones are the *Pauli matrixes*.
  – These form a basis for an 2x2 hermitian matrix.

Examples

- For example consider:
  - linearly polarised waves in the horisontal plane $e_M^\alpha = [1, 0]$

$$p^{\alpha\beta} = e_M^{\alpha\,*} e_M^\beta = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + 1\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right)$$

$$\rightarrow \{q, u, v\} = \{1,0,0\}$$

  - rotate linear polarization by 45°, $e_M^\alpha = [1, 1]2^{1/2}$

$$p^{\alpha\beta} = e_M^{\alpha\,*} e_M^\beta = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + 1\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right)$$

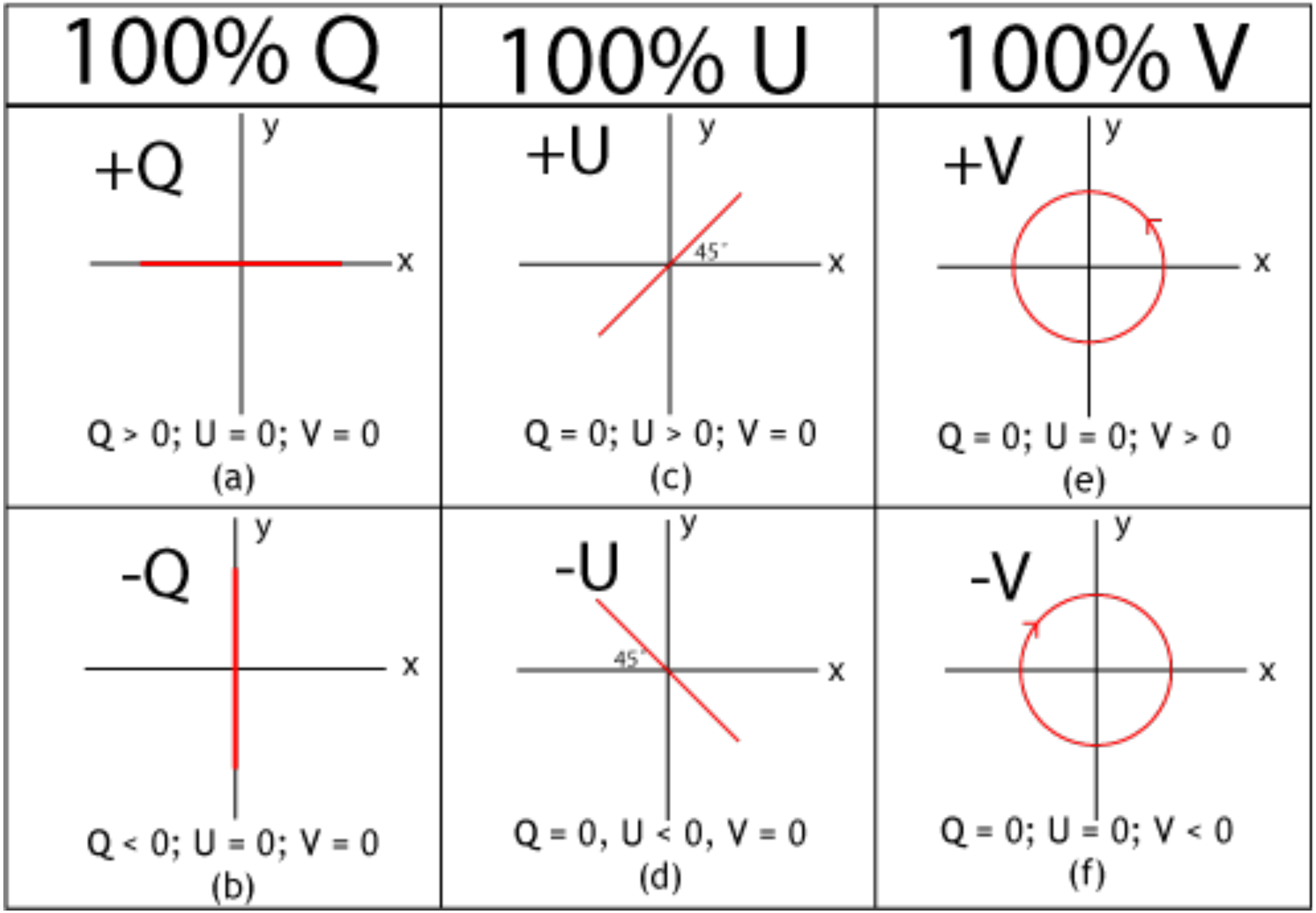
$$\rightarrow \{q, u, v\} = \{0,1,0\}$$

  - right hand circularly polarised waves, $e_M^\alpha = [1, -i] / 2^{1/2}$

$$p^{\alpha\beta} = e_M^{\alpha\,*} e_M^\beta = \frac{1}{2}\begin{bmatrix}1\\i\end{bmatrix}\begin{bmatrix}1 & -i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -i\\i & 1\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + 1\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\right)$$

$$\rightarrow \{q, u, v\} = \{0,0,1\}$$

Table of ideal polarisations



The Stokes vector

- Using unit- and Pauli-matrixes, we define $\tau_j^{\alpha\beta}$ as:

$$\tau_1^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_2^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_3^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_4^{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- These can also be used to express the intensity tensor:

$$I^{\alpha\beta} = \frac{1}{2}\left[I\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Q\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + U\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + V\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = \frac{1}{2}\begin{bmatrix} I+Q & U-iV \\ U+iV & I-Q \end{bmatrix}$$

  - The four parameters are called the Stokes parameter $\{ I, Q, U, V \}$
  - The Stokes vector is similarly defined as $S_A\big|_{A=\{1,2,3,4\}} = [I, Q, U, V]$

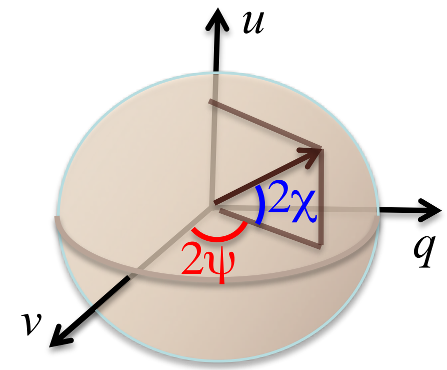
- Using index notation the intensity matrix and the Stokes vector are related by:

$$I^{\alpha\beta} = \frac{1}{2}\tau_A^{\alpha\beta} S_A \quad \text{with inverse:} \quad S_A = \tau_A^{\alpha\beta} I^{\alpha\beta}$$

  - The matrixes $\tau_j^{\alpha\beta}$, defines a transformation between hermitian 2x2 matrixes and real 4-vectors

Poincare sphere

- Define the *degree of polarisation*: $r = \sqrt{q^2 + u^2 + v^2}$
- Consider the normalised vector $\{ q/r\ ,\ u/r\ ,\ v/r \}$ (polarised fraction)
  - since this vector is real and normalised it will represent points on a sphere, the so called *Poincare sphere*



Poincare sphere

- Thus, any transverse wave field can be described by
  - a point on the Poincare sphere
  - a degree of polarization, r
- A polarizing element induces a motion on the sphere
  - e.g. passing though a birefringent crystal traces a *circle* :
    - Birefringence rotates in a vertical plane
    - Faraday rotation rotates in a horisontal plane
  - Here is may be useful to polar coordinates (ψ,χ).

The polarization tensor for unpolarized waves (1)

- What are the Stokes parameters for unpolarised waves?
  - Let the e_M^1 and e_M^2 be independent stochastic variable

$$p^{\alpha\beta} = < \begin{pmatrix} e_M^1 \\ e_M^2 \end{pmatrix} \begin{pmatrix} e_M^1 & e_M^2 \end{pmatrix}^* > = \begin{bmatrix} < e_M^1 e_M^{1\,*} > & < e_M^1 e_M^{2\,*} > \\ < e_M^2 e_M^{1\,*} > & < e_M^2 e_M^{2\,*} > \end{bmatrix}$$

  - Since e_M^1 and e_M^2 are uncorrelated the offdiagonal term vanish

$$p^{\alpha\beta} = \begin{bmatrix} <| e_M^1 |^2> & 0 \\ 0 & <| e_M^2 |^2> \end{bmatrix}$$

  - The vector $\mathbf{e}_M$ is normalised: $\left|e_M^1\right|^2 + \left|e_M^2\right|^2 = 1$
  - By symmetry (no physical difference between e_M^1 and e_M^2)

$$\left|e_M^1\right|^2 = \left|e_M^2\right|^2 = 1/2$$

  - the polarization tensor then reads

$$p^{\alpha\beta} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  - i.e. unpolarised have $\{q,u,v\} = \{0,0,0\}!$

The polarization tensor for unpolarized waves (2) ★

- Alternative derivation; polarization vector for unpolarized waves
  - Note first that the polarization vector is normalised

$$\left|\mathbf{e}_M\right|^2 = \left|e_M^1\right|^2 + \left|e_M^2\right|^2 = 1 \sim \cos^2(\theta) + \sin^2(\theta)$$

  - the polarization is complex and stochastic: $\begin{pmatrix} e_M^1 \\ e_M^2 \end{pmatrix} = \begin{pmatrix} e^{i\phi_1}\cos(\theta) \\ e^{i\phi_2}\sin(\theta) \end{pmatrix}$
    - where θ, ϕ_1 and ϕ_2 are uniformly distributed in $[0,2\pi]$
- The corresponding polarization tensor

$$p^{\alpha\beta} = < \begin{pmatrix} e_M^1 \\ e_M^2 \end{pmatrix} \begin{pmatrix} e_M^1 & e_M^2 \end{pmatrix}^* > = < \begin{pmatrix} e^{i\phi_1 - i\phi_1}\cos(\theta)\cos(\theta) & e^{i\phi_1 - i\phi_2}\cos(\theta)\sin(\theta) \\ e^{i\phi_2 - i\phi_1}\sin(\theta)\cos(\theta) & e^{i\phi_2 - i\phi_2}\sin(\theta)\sin(\theta) \end{pmatrix} >$$

  - here the average is over the three random variables θ, ϕ_1 and ϕ_2

$$p^{\alpha\beta} = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \begin{pmatrix} \cos^2(\theta) & e^{i\phi_1 - i\phi_2}\cos(\theta)\sin(\theta) \\ e^{i\phi_2 - i\phi_1}\sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  - i.e. unpolarised have $\{q,u,v\} = \{0,0,0\}$!

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- **Muller calculus**; matrix formulation for the transmission of partially polarized waves
  - General theory for dispersive media
  - Optical components

Weakly anisotropic media

- In weakly anisotropic media the wave equation can be rewritten on a form suitable for studying the wave polarisation.
- Write the weakly anisotropic transverse response as

$$K^{\alpha\beta} = {n_0}^2\delta^{\alpha\beta} + \Delta K^{\alpha\beta}$$

  - where $\Delta K^{\alpha\beta}$ is a small perturbation
- The wave equation

$$\left(n^2\delta^{\alpha\beta} - K^{\alpha\beta}\right)E^\beta = 0 \quad \Rightarrow \quad \left(n^2 - {n_0}^2\right)E^\alpha = \Delta K^{\alpha\beta}E^\alpha$$

  - when ΔK_{ij} is a small, the 1st order dispersion relation reads: $n^2 \approx {n_0}^2$
  - the left hand side can then be expanded to give

small, <<1

$$n^2 - n_0^2 = (n - n_0)(n + n_0) = (n - n_0)n_0\left[2 + \frac{(n - n_0)}{n_0}\right] \approx 2n_0(n - n_0)$$

$$2n_0(n - n_0)E^\alpha \approx \Delta K^{\alpha\beta}E^\alpha$$

The wave equation in Jones calculus

- Inverse Fourier transform, when $k_0 = \omega n_0 / c$:

$$2k_0(k - k_0)E^\alpha \approx \frac{\omega^2}{c^2}\Delta K^{\alpha\beta}E^\alpha \Leftrightarrow 2k_0(-i\frac{\partial}{\partial x} - k_0)E^\alpha \approx \frac{\omega^2}{c^2}\Delta K^{\alpha\beta}E^\alpha$$

- Factor our the eikonal with wave number k_0 :

$$E^\alpha = E_0^\alpha(x)\exp\left(ik_0 x\right)$$

- The wave equation can then be simplified

$$(-i\frac{\partial}{\partial x} - k_0)E_0^\alpha(x)\exp\left(ik_0 x\right) \approx \frac{\omega^2}{2k_0 c^2}\Delta K^{\alpha\beta}E_0^\alpha(x)\exp\left(ik_0 x\right)$$

$$\frac{dE_0^\alpha}{dx} \approx i\frac{\omega}{2n_0 c}\Delta K^{\alpha\beta}E_0^\beta$$

$$\Rightarrow$$

$$E_{out}^\alpha = J^{\alpha\beta}E_{in}^\beta \; ; \; J^{\alpha\beta} \approx \exp\left[i\frac{\omega}{2n_0 c}\Delta K^{\alpha\beta}x\right]$$

The differential transfer equation in the Jones calculus!

(We will use this relation in the next lecture)

The wave equation as an ODE

- Wave equation for the intensity tensor:

$$\frac{dI^{\alpha\beta}}{dx} = \underbrace{\frac{d}{dx} < E^{\alpha} E^{\beta^*} >}_{} = \ldots = \frac{i\omega}{2cn_0}\left(\Delta K^{\alpha\rho}\delta^{\beta\sigma} - \Delta K^{\beta\sigma^*}\delta^{\alpha\rho}\right)I^{\rho\sigma}$$

from prev. page:

$$\frac{dE_0^{\alpha}}{dx} \approx i\frac{\omega}{2n_0 c}\Delta K^{\alpha\beta}E_0^{\beta}$$

- Rewrite it in terms of the Stokes vector: $S_A = \tau_A^{\alpha\beta} I^{\alpha\beta}$

$$\frac{dS_A}{dx} = \left(\rho_{AB} - \mu_{AB}\right)S_B \qquad \begin{cases} \rho_{AB} = \dfrac{i\omega}{4cn_0}\left(\tau_A^{\beta\alpha}\Delta K^{H,\alpha\rho}\tau_B^{\rho\beta} - \tau_B^{\rho\sigma}\Delta K^{H,\sigma\beta}\tau_A^{\beta\rho}\right) \\ \mu_{AB} = -\dfrac{i\omega}{4cn_0}\left(\tau_A^{\beta\alpha}\Delta K^{A,\alpha\rho}\tau_B^{\rho\beta} + \tau_B^{\rho\sigma}\Delta K^{A,\sigma\beta}\tau_A^{\beta\rho}\right) \end{cases}$$

  - we may call this the *differential formulation of Muller calculus*
  - symmetric matrix ρ_{AB} describes non-dissipative changes in polarization
  - and the antisymmetric matrix μ_{AB} describes dissipation (absorption)

The wave equation as an ODE

- The ODE for S_A has the analytic solution (cmp to the ODE $y' = ky$)

$$S_A(x) = \left[\delta_{AB} + \left(\rho_{AB} - \mu_{AB}\right)x + 1/2\left(\rho_{AC} - \mu_{AC}\right)\left(\rho_{CB} - \mu_{CB}\right)x^2 + \ldots\right]S_B(0)$$

  - cmp with Taylor series for exponential

$$S_A(x) = M_{AB}S_B(0) \text{ , where } M_{AB} = \exp\left[\left(\rho_{AB} - \mu_{AB}\right)x\right]$$

- Here M_{AB} is called the *Muller matrix*
  - M_{AB} represents entire optical components / systems
  - This is a *component based Muller calculus*

Examples of Muller matrixes

Some common Muller matrixes:

Linear polarizer (Horizontal Transmission)

$$M_{AB}^{L,H} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Linear polarizer (45^o transmission)

$$M_{AB}^{L,45} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Quarter wave plate (fast axis horizontal)

$$M_{AB}^{Q,H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Attenuating filter (30% Transmission)

$$M_{AB}^{Att}(0.3) = 0.3\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is the Muller matrix for Faraday rotation?

Examples of Muller matrixes

- In optics it is common to connect a series of optical elements
- consider a system with:
  - a linear polarizer and
  - a quarter wave plate

$$S_A^{out} = M_{AB}^{Q,H} M_{BC}^{L,45} S_C^{in}$$

- Insert unpolarised light, $S_A{}^{in}=[1,0,0,0]$
  - **Step 1**: Linear polariser transmit linearly polarised light

$$S^{step1} = M_{BC}^{L,45}\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T = \begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix}^T$$

  - **Step 2**: Quarter wave plate transmit circularly polarised light

$$S^{out} = M^{Q,H}\begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix}^T = \begin{bmatrix}1 & 0 & 0 & -1\end{bmatrix}^T$$