Lecture 8
OFDM and Channel Capacity

M. Xiao
CommTh/EES/KTH

Lecture 8: OFDM and Channel Capacity
Advanced Digital Communications (EQ2410)[1]

M. Xiao
CommTh/EES/KTH

Wednesday, Feb. 17, 2016
10:00-12:00, B23

[1] Textbook: U. Madhow, *Fundamentals of Digital Communications*, 2008

Notes

Lecture 8
OFDM and Channel Capacity

M. Xiao
CommTh/EES/KTH

Overview

Lecture 7

- Characteristics of wireless channels
- Performance for fading channels
- Diversity

Lecture 8: OFDM and Channel Capacity

Notes

Motivation

Applications of OFDM (orthogonal frequency division multiplexing)

- Digital subscriber line (DSL)
- Digital video broadcast (DVB-S/T)
- WLAN (IEEE 802.11)
- WIMAX (IEEE 802.16)
- LTE/IMT-Advanced (4G/4.5G)

Conventional signaling with one carrier

- Effective pulse: $x(t) = (g_T \star g_C \star g_R)(t)$
- Received signal: $y(t) = \sum_k b[k]x(t-kT)$ (plus noise)
- ISI avoidance (Nyquist criterion): The waveforms $x(t-kT)$ must be orthogonal.

→ Design of $g_R(t)$ and $g_T(t)$; difficult to achieve!

Notes

Lecture 8
OFDM and Channel Capacity
M. Xiao
CommTh/EES/KTH

Concept OFDM

Interesting observation

Theorem 8.3.1 *Consider a linear time-invariant channel with impulse response $g_C(t)$ and transfer function $G_C(f)$. Then the following statements are true:*

(a) The complex exponential waveform $e^{j2\pi ft}$ is an eigenfunction of the channel with eigenvalue $G_C(f)$. That is,

$$e^{j2\pi ft} * g_C(t) = G_C(f)e^{j2\pi ft}.$$

(b) Complex exponentials at different frequencies are orthogonal.

[Madhow, *Fundamentals of Digital Communication*, 2008]

→ Orthogonality is preserved after transmission through the channel!

OFDM system

- Discrete set of N carriers over a symbol interval of finite length T

$$u(t) = \sum_{n=0}^{N-1} B[n]e^{j2\pi f_n t} I_{[0,T]}(t) = \sum_{n=0}^{N-1} B[n]p_n(t)$$

- The symbols $B[n]$ are mapped to the carriers.

Notes

Concept OFDM

What about the orthogonality?

- Orthogonality for two sub-carriers $p_n(t)$ and $p_m(t)$:

$$\langle p_n, p_m \rangle = \int_0^T e^{j2\pi f_n t} e^{-j2\pi f_m t} dt = \frac{e^{j2\pi(f_n - f_m)T} - 1}{j2\pi(f_n - f_m)}$$

$\rightarrow$ orthogonal if $(f_n - f_m)T =$ non-zero integer; for example $f_n = n/T$.

- Fourier transform of $p_n(t)$:

$$P_n(f) = T \cdot \text{sinc}((f - f_n)T)e^{-j\pi(f-f_n)T}$$

$\rightarrow$ decays quickly as $|f - f_n|$ takes on values of the order of k/T

- If $T \gg T_m \Rightarrow 1/T \ll B_m$; i.e., each sub-carrier sees an approximately constant channel (frequency-flat fading), and we have

$$Q_n(f) = G_C(f)P_n(f) \approx G_C(f_n)P_n(f).$$

$\rightarrow$ Orthogonality is preserved after transmission through the channel.

Notes

Implementation OFDM

- Transmitted OFDM waveform with $f_n = n/T$ (one OFDM symbol)

$$u(t) = \sum_{n=0}^{N-1} B[n]p_n(t) = \sum_{n=0}^{N-1} B[n]e^{j2\pi nt/T} I_{[0,T]}(t)$$

- Sampled OFDM signal with $T_s = 1/W = T/N$

$$u(kT_s) = \sum_{n=0}^{N-1} B[n]e^{j2\pi nk/N}$$

$\rightarrow$ Inverse DFT of $B[n]$; i.e., $u(kT_S) =$ I-DFT$(B[n]) = b[k]$
$\rightarrow$ Efficient implementation with FFT/IFFT for $N = 2^i$.

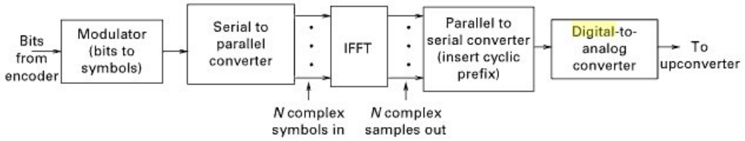
- OFDM demodulation with FFT (without considering the channel)

$$B[n] = \frac{1}{N}\sum_{k=0}^{N-1} b[k]e^{-j2\pi nk/N} = FFT(b(k)) = FFT(u(kT_s))$$

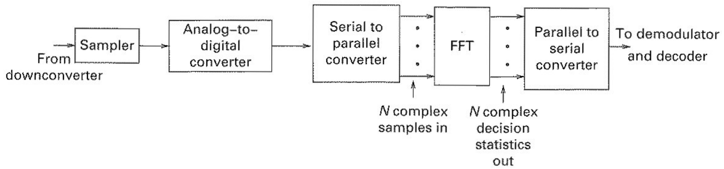
Notes

Implementation OFDM

Transmitter



Receiver



[Madhow, *Fundamentals of Digital Communication*, 2008]

Notes

Cyclic Prefix

- Received signal with effective pulse $p(t)$ (incl. D/A conversion at the transmitter, physical channel, receive filter; noise-free case)

$$v(t) = \sum_{k=0}^{N-1} b[k]p(t - kT_s) \quad \text{and} \quad v[m] = \sum_{k=0}^{N-1} b[k]h[m-k]$$

  with $h[l] = p(lT_s)$, the sampled impulse response of length L.

- Problem
  - Linear convolution of $b[k]$ and $h[k]$
  - To preserve the property $V[n] = H[n]B[n]$, with $H[n] = DFT_N(h[l])$, a cyclic convolution is required.

- Solution: cyclic prefix
  - By appending the last $L-1$ symbols $b[N-L+1], \dots, b[N-1]$ as a prefix to the symbols $b[n]$ a linear convolution of the channel with
    $$b[N-L+1], \dots, b[N-1], b[0], \dots, b[N-1]$$
    becomes a cyclic convolution of the channel and $b[0], \dots, b[N-1]$.
  - After sampling the received signal, removing the length-L cyclic prefix, and applying the length-N DFT to received samples we get
    $$Y[n] = H[n]B[n] + N[n]$$

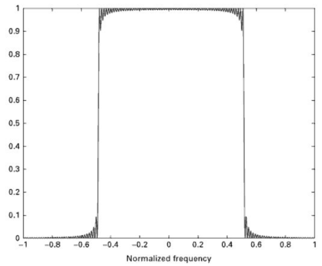
$\rightarrow$ OFDM transforms a frequency-selective channel into N narrowband channels with fading coefficients $H[n]$ (no additional equalization).

Notes

Lecture 8
OFDM and Channel Capacity

M. Xiao
CommTh/EES/KTH

PSD of OFDM Signals



[Madhow, *Fundamentals of Digital Communication*, 2008]

$$
\begin{aligned}
S_u(f) &= \sum_{n=0}^{N-1} \mathrm{E}[|B[n]|^2] \frac{|P_n(f)|^2}{T} \\
&= T \sum_{n=0}^{N-1} \sigma_B^2[n] |\mathrm{sinc}((f-f_n)T)|^2
\end{aligned}
$$

Notes

Lecture 8
OFDM and Channel Capacity
M. Xiao
CommTh/EES/KTH

Peak-to-Average Ratio (PAR)

- $b[k]$ is the sum of a large number of independent terms
  $\rightarrow$ central limit theorem: $b[k] \sim N(0, \sigma_b^2)$
- Problem in OFDM: high instantaneous power $P[k] = |b[k]|^2$ can occur $\rightarrow$ problem with linearity of amplifiers.
- Peak-to-Average Ratio

$$
PAR = \frac{\max\limits_{0 \le k \le N-1} P[k]}{\frac{1}{N} \sum\limits_{k=0}^{N-1} P[k]}
$$

- Methods to reduce PAR
  - power reduction by approximately PAR dB
  - insert different phase shifts in each of the sub-carriers
  - modulate dummy symbols which are selected in order to reduce PAR
  - insert redundancy into data on carriers by expanding the constellation
  - $\rightarrow$ loss of efficiency (power, rate,...)

Notes

Capacity

- OFDM transforms a frequency selective channel into N parallel narrowband fading channels.
- In the following, K parallel Gaussian channels with

$$Y_k = h_k X_k + Z_k, \quad \text{with } k \in \{1, \ldots, K\},$$

the channel gain h_k, $Z_k \sim CN(0, N_k)$, and $\mathrm{E}[|X_k|^2] = P_k$.

- Assuming independence of the channels, the sum rate/capacity for a given power allocation $\mathbf{P} = [P_1, \ldots, P_K]$ is given as

$$C(\mathbf{P}) = \sum_{k=1}^{K} \log_2 \left(1 + \frac{|h_k|^2 P_k}{N_k}\right)$$

Optimal power allocation: water filling

Notes

Water Filling

- Goal: maximize $C(\mathbf{P})$ subject to the constraint $\sum_{k=1}^{K} P_k \leq P$
- Maximize the Lagrangian

$$J(\mathbf{P}) = C(\mathbf{P}) - \lambda \left(\left[\sum_{k=1}^{K} P_k \right] - P \right) = \sum_{k=1}^{K} \log_2 \left(1 + \frac{|h_k|^2 P_k}{N_k}\right) - \lambda \left(\left[\sum_{k=1}^{K} P_k \right] - P \right)$$

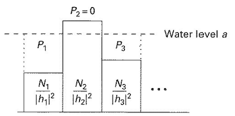
- By setting the first derivative of $J(\mathbf{P})$ to zero we get

$$P_k = a - \frac{N_k}{|h_k|^2}$$

- By choosing a such that the power constraint is fulfilled we get the water-filling solution

$$P_k = \left[a - \frac{N_k}{|h_k|^2} \right]^+$$

with $[x]^+ = x$, if $x > 0$, and $[x]^+ = 0$, else.



→ for channels with $N_k/|h_k|^2 > a$ we get $P_k = 0$.
→ If power is limited, the power is allocated to the good channels first.

Notes

Capacity OFDM

- k-th sub-carrier

$$Y_k = H(f_k)X_k + Z_k$$

with $Z_k \sim CN(0, S_n(f_k)\Delta f)$, $\mathrm{E}[|X_k|^2] = S_s(f_k)\Delta f$, the PSDs $S_s(f)$ and $S_n(f)$ of the signal and the noise, and the bandwidth of the sub-carriers Δf.

- Sum rate (capacity) for an OFDM system

$$R = \sum_k \Delta f \log_2 \left(1 + \frac{|H(f_k)|^2 S_s(f_k)}{S_n(f_k)}\right)$$

- Capacity and power constraint for a frequency-selective channel (follows for $\Delta f \to 0$)

$$C = \int_{-W/2}^{W/2} \log_2 \left(1 + \frac{|H(f)|^2 S_s(f)}{S_n(f)}\right) df \quad \text{and} \quad \int_{-W/2}^{W/2} S_s(f) df = P$$

with the water-filling solution $S_s(f) = [a - S_n(f)/|H(f)|^2]^+$

- For $N \to \infty$ and $\Delta f \to 0$ together with the optimal power allocation, OFDM becomes capacity achieving.

Notes

Resource Allocation

1. Find the optimal power allocation maximizing the sum rate under the given power constraint.
2. Choose the modulation and coding such that the rate can be reliably achieved.

Notes