KTH ROYAL INSTITUTE
OF TECHNOLOGY

ED2210

Lecture 1: Energy continuity

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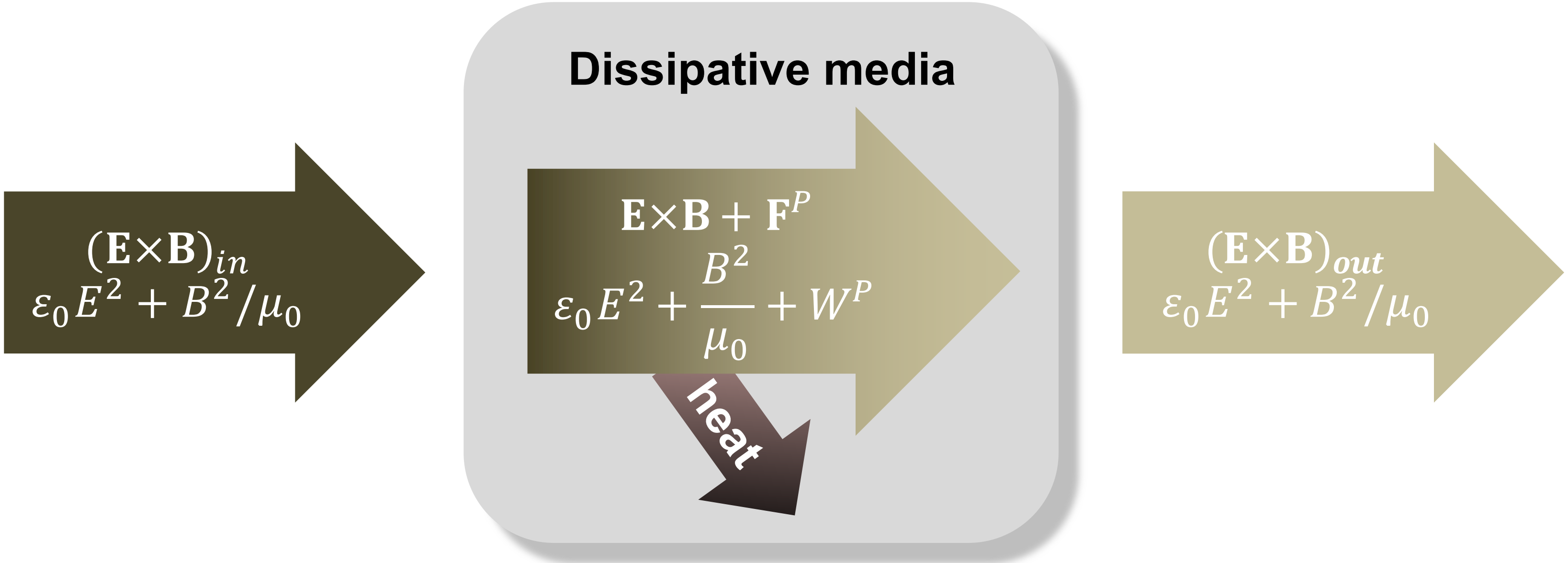
Energy continuity in a media

Energy continuity concerns the conservation of energy.

- In dispersive media the energy of the wave includes both electric, magnetic as well as *particle (kinetic/mechanical) energy*.
- Similarly the energy flux includes both an electro-magnetic pointing flux $\mathbf{E}\times\mathbf{B}$ and a *kinetic energy flux*, $\mathbf{F}_{kinetic}$.



Energy continuity in a media

- Pure electro-magnetic energy (see Lecture 1):

$$\frac{\partial}{\partial t}(W^E + W^M) + \nabla \cdot (\mathbf{E}\times\mathbf{B}) = -\mathbf{E}\cdot\mathbf{J}$$

  - where W^E and W^M are the electric and magnetic energy densities, $\mathbf{E}\times\mathbf{B}$ is the Poynting flux, and $\mathbf{J}\cdot\mathbf{E}$ is the work by $\mathbf{E}$ in the current.
- In a media, particles are pushed by the E-field, giving them a kinetic wave energy, including two parts:
  - thermal energy (unordered motion), W^{th}
  - wave energy (ordered motion), W^P

$$\frac{\partial}{\partial t}W^P + \nabla \cdot \mathbf{F}^P = (\mathbf{E}\cdot\mathbf{J})_{reactive}$$ *Kinetic wave energy*

$$\frac{\partial}{\partial t}W^{th} + \nabla \cdot \mathbf{F}^{th} = (\mathbf{E}\cdot\mathbf{J})_{dissipative}$$ *Thermal energy (heat)*

- Where $\mathbf{F}^P$ is the flux of kinetic wave energy.

$$(\mathbf{E}\cdot\mathbf{J})_{dissipative} = E_i\sigma_{ij}^H * E_j \qquad (\mathbf{E}\cdot\mathbf{J})_{reactive} = E_i\sigma_{ij}^A * E_j$$

Energy continuity in a media

The total wave energy; the sum of electromagnetic and particle energy:

$$\frac{\partial}{\partial t}(W^E + W^M + W^P) + \nabla \cdot (\mathbf{E}\times\mathbf{B} + \mathbf{F}^P) = -(\mathbf{J}\cdot\mathbf{E})_{dissipative}$$

Terms specific to dispersive media

or

$$\frac{\partial}{\partial t}W^{wave} + \nabla \cdot \mathbf{F}^{wave} = -\gamma W^{wave} \quad (1)$$

Here γ is the dissipative damping rate.

Below we'll study:

- How to calculate W^E
- The ratio W^M/W^E and W^E/W are relate to the dielectric tensor
- How the energy flux, $\mathbf{F}^{wave}$, is related to the group velocity
- How to describe temporal and spatial damping using (1).

Outline

- Representations of a wave mode with given $\omega_M(\mathbf{k})$ and $\mathbf{e}_M(\mathbf{k})$
- Evaluation of averaged energy density and dissipative work
  - Infinite integral has to be handled like an integral of a generalised function
  - New concept: Phase-space energy density
    - Expressions for: W^E, W^M, W^P and $(\mathbf{J}\cdot\mathbf{E})_{dissipative}$

- Expand the dispersion equation $\det(\Lambda) = 0$
  - Temporal and spatial damping
  - Continuity equation for wave energy
  - Energy flux is related to the group velocity!
    - Expressions for $\mathbf{F}^P$ and $\mathbf{E}\times\mathbf{B}$

Representations of a wave mode

A good representation will be needed to simplify the upcoming algebra

- A wave mode is defined by
  - dispersion relation: $\omega = \omega_M(\mathbf{k})$, no other frequencies possible!
  - eigenvector: $\mathbf{e}_M(\mathbf{k})$
- Simplified representation of the vector potential:

$$\mathbf{A}(\omega, \mathbf{k}) = \hat{a}_M(\mathbf{k})\mathbf{e}_M(\mathbf{k})2\pi\delta(\omega - \omega_M(\mathbf{k}))$$

  - Scalar representation of waves: $\hat{a}_M(\mathbf{k})$
- Temporal gauge:

$$\mathbf{E}(\omega, \mathbf{k}) = -i\omega\mathbf{A}(\omega, \mathbf{k})$$
$$\mathbf{B}(\omega, \mathbf{k}) = i\mathbf{k}\times\mathbf{A}(\omega, \mathbf{k})$$

Representations of a wave mode

- The vector potential also has to satisfy the reality condition
- I.e. since $\mathbf{A}(t,\mathbf{x})$ is real then $\mathbf{A}(\omega,\mathbf{k}) = \mathbf{A}^*(-\omega,-\mathbf{k})$.
  $\Longrightarrow\ \hat{a}_M(\mathbf{k})\mathbf{e}_M(\mathbf{k})2\pi\delta\big(\omega-\omega_M(\mathbf{k})\big) = \hat{a}_M^*(-\mathbf{k})\,\mathbf{e}_M^*(-\mathbf{k})2\pi\delta(-\omega-\omega_M(-\mathbf{k}))$

Note: This condition should be valid for any amplitude $\hat{a}_M(\mathbf{k})$

- Thus for every mode $\{\omega_M(\mathbf{k}),\mathbf{e}_M(\mathbf{k})\}$ there is a mode $\{-\omega_M(-\mathbf{k}),\mathbf{e}_M^*(-\mathbf{k})\}$
- To see why consider a scalar plane wave:

$$A\cos(kx-\omega t) = \frac{1}{2}(A\exp(-i\omega t + ikx) + A^*\exp(i\omega t - ikx))$$

  - Plane waves includes two terms: $\{\omega,\mathbf{k}\}$ and $\{-\omega,-\mathbf{k}\}$
  - The dielectric response of the terms are related:

$$\omega_M(\mathbf{k}) = -\omega_M(-\mathbf{k})\ \ \&\ \ \mathbf{e}_M(\mathbf{k}) = \mathbf{e}_M^*(-\mathbf{k})$$

New representation that always satisfy the reality condition:

$$\mathbf{A}(\omega,k) = a_M(\mathbf{k})\mathbf{e}_M(\mathbf{k})2\pi\delta\big(\omega-\omega_M(\mathbf{k})\big) + a_M^*(-\mathbf{k})\,\mathbf{e}_M^*(-\mathbf{k})2\pi\delta(\omega+\omega_M(-\mathbf{k}))$$

Phase-space energy density

- Wave energy is a quadratic quantity, e.g. the electric energy density: $W^E(t, \mathbf{x}) = \varepsilon_0 |\mathbf{E}(t, \mathbf{x})|^2/2$
- The Fourier transform of this expression gives a convolution!
  - Which makes further mathematical analysis difficult
- Instead we'll here consider called a *phase-space density*.
  - a density in both x-space and k-space.
- Here is an "almost rigorous" derivation…
- Calculate the mean electric energy density in a volume V and over a time interval $[-T/2, T/2]$:

$$< W >= \frac{1}{TV} \iiint_V d^3x \int_{-T/2}^{T/2} dt \frac{1}{2} \varepsilon_0 |\mathbf{E}(t, \mathbf{x})|^2$$

Phase-space energy density

- Take the limit when $T \to \infty$ and when V cover all of space.

$$\lim_{T\to\infty, V\to\infty} < W > = \lim_{T\to\infty, V\to\infty} \frac{1}{TV} \iiint_V d^3x \int_{-T/2}^{T/2} dt \frac{1}{2} \varepsilon_0 |\mathbf{E}(t,\mathbf{x})|^2$$

- These infinite integrals can only be evaluated as generalised function, solve them using a truncated E-field (see lecture 2)

$$\mathbf{E}_{T,V}(t,\mathbf{x}) = \begin{cases} \mathbf{E}(t,\mathbf{x}), t \in [-T/2, T/2] \text{ and } \mathbf{x} \in V \\ 0, \quad \text{otherwise} \end{cases}$$

$$\lim_{T\to\infty, V\to\infty} < W > = \lim_{T\to\infty, V\to\infty} \frac{1}{TV} \iiint_{R^3} d^3x \int_{-\infty}^{\infty} dt \frac{1}{2} \varepsilon_0 \left|\mathbf{E}_{T,V}(t,\mathbf{x})\right|^2$$

- Use the power theorem:

$$= \lim_{T\to\infty, V\to\infty} \frac{1}{TV} \iiint_{R^3} \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{2} \varepsilon_0 \left|\mathbf{E}_{T,V}(\omega,\mathbf{k})\right|^2$$

Phase-space energy density

- Use the simplified wave-mode representation:

$$\mathbf{A}_M(\omega, \mathbf{k}) = a_M(\mathbf{k})\mathbf{e}_M(\mathbf{k})2\pi\delta\big(\omega - \omega_M(\mathbf{k})\big) + a_M^*(-\mathbf{k})\ \mathbf{e}_M^*(-\mathbf{k})2\pi\delta(\omega + \omega_M(-\mathbf{k}))$$

- But here we need a truncated version!

$$\begin{cases} \mathbf{e}_M(\mathbf{k}) \rightarrow \mathbf{e}_M(\mathbf{k}) \\ a_M(\mathbf{k}) \rightarrow a_{M,V}(\mathbf{k}) \\ \delta\big(\omega - \omega_M(\mathbf{k})\big) \rightarrow \delta_T\big(\omega - \omega_M(\mathbf{k})\big) \equiv \dfrac{T}{2\pi}\dfrac{\sin\big[\big(\omega - \omega_M(\mathbf{k})\big)T/2\big]}{\big(\omega - \omega_M(\mathbf{k})\big)T/2} \end{cases}$$

- Note: ${\mathbf{A}_M}^2$ has four terms; the cross-terms vanishes and the other two are the same

$$\lim_{V\to\infty} < W > = \lim_{V\to\infty} \frac{1}{V} \iiint_{R^3} \frac{d^3k}{(2\pi)^3} \varepsilon_0 \left|a_{M,V}(\mathbf{k})\right|^2 \underbrace{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{\omega^2}{T}\left[2\pi\delta_T\big(\omega - \omega_M(\mathbf{k})\big)\right]^2}_{= \omega_M(\mathbf{k})^2}$$

$$\lim_{V\to\infty} < W > = \lim_{V\to\infty} \iiint_{R^3} \frac{d^3k}{(2\pi)^3} W_{M,V}^E(\mathbf{k}), \qquad \text{where } W_{M,V}^E(\mathbf{k}) \equiv \frac{1}{V}\varepsilon_0\left|\omega_M(\mathbf{k})a_{M,V}(\mathbf{k})\right|^2$$

Wave-mode representation of energy

Thus:

$$W^E_{M,V}(\omega,\mathbf{k}) = \frac{1}{V}\varepsilon_0\left|\omega_M(\mathbf{k})a_{M,V}(\mathbf{k})\right|^2$$

is the *electric phase-space energy density*; a density in both…

- $\mathbf{k}$-space; the actual energy is related to an integral over $\mathbf{k}$
- $\mathbf{x}$-space; from the definition of "mean energy density" (and $1/V$)

$W^E_{M,V}(\omega,\mathbf{k})$ is constructed from a Fourier transform of a truncated function.

- $W^E_{M,V}(\omega,\mathbf{k})$ is a generalised function!
- E.g. for plane waves, $a_M(\mathbf{x}) \sim \exp(ik_{0,z}z)$
  - when V is a cube with sides of length L
  - the Fourier transform of the truncated function reads:

$$a_{M,V}(\mathbf{k}) \sim \delta_L(k - k_{0,z}) \equiv \frac{L}{2\pi}\frac{\sin[(k-k_{0,z})L/2]}{(k-k_{0,z})L/2}$$

Magnetic phase space density

Similarly we may derive a *magnetic phase space energy density*:

$$W_M^M(\mathbf{k}) = \frac{1}{\mu_0 V}|\mathbf{k}\times\mathbf{e}_M(\mathbf{k})a_M(\mathbf{k})|^2 = \frac{k^2c^2}{\omega_M(\mathbf{k})^2}\left|1-|\boldsymbol{\kappa}\cdot\mathbf{e}_M(\mathbf{k})|\right|^2 W_M^E(\mathbf{k})$$

where $\boldsymbol{\kappa} = \mathbf{k}/|\mathbf{k}|$.

No magnetic energy for longitudinal waves

The *kinetic phase-space energy density*, $W_M^P(\mathbf{k})$, is kinetic energy of particles of the wave; when pushed around by the EM-field.

There is no simple formula describing $W_M^P(\mathbf{k})$.

The *total phase-space energy density* of a wave-mode is then

$$W_M(\mathbf{k}) = W_M^E(\mathbf{k}) + W_M^M(\mathbf{k}) + W_M^P(\mathbf{k})$$

To characterise the kinetic (particle) energy it is useful to define

$$R_M(\mathbf{k}) := \frac{W_M^E(\mathbf{k})}{W_M(\mathbf{k})}$$

Work by an electric field on a current

Use a similar procedure for the work performed by an electric field on a current:

$$\lim_{V\to\infty} < \mathbf{E}\cdot\mathbf{J} > = \lim_{V\to\infty} \iiint_{R^3} \frac{d^3k}{(2\pi)^3} Q_{M,V}(\mathbf{k})$$

Anti-hermitian part polarisation tensor: $J_i = \alpha_{ij} A_j$

$$Q_{M,V}(\mathbf{k}) \equiv i2\omega_M(\mathbf{k}) \frac{|a_{M,V}(\mathbf{k})|^2}{V} e^*_{M,i}(\mathbf{k})\alpha^A_{ij}(\mathbf{k})e_{M,j}(\mathbf{k})$$

Thus, Q_M is the work per phase-space $\{\mathbf{x},\mathbf{k}\}$ volume.
Alternatively, the dissipative power transfer from wave to the media.

$Q_{M,V}$ is proportional to the electric energy density, $W^E_M(\mathbf{k})$; identify *damping rate*, γ :

$$Q_{M,V}(\mathbf{k}) \equiv -\gamma W_M(\mathbf{k}) = -\gamma R_M(\mathbf{k}) W^E_M(\mathbf{k})$$

$$\gamma = -i2\omega_M(\mathbf{k}) R_M(\mathbf{k}) \left\{ e^*_{M,i}(\mathbf{k}) K^A_{ij}(\mathbf{k}) e_{M,j}(\mathbf{k}) \right\}$$

***Note**: same as in Lec. 6*

Damping in time and space

- Energy continuity equation

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{P} = (\mathbf{E} \cdot \mathbf{J})_{disspation}$$

  - Here W is the total energy density, inc. electric magnetic and kinetic energy, of the wave
  - $\mathbf{P}$ is the energy flux, inc. Pointing flux and kinetic energy flux
    - We will show that: $\mathbf{P} = W\mathbf{v}_g$, where $\mathbf{v}_g$ is the group velocity
- Initial wave problem with damping; temporal damping, $E^2 \sim e^{-2\omega_I t}$

$$\frac{\partial W}{\partial t} = -2\omega_I W = (\mathbf{E} \cdot \mathbf{J})_{disspation}$$

- Boundary value problem with damping: spatial damping, $E^2 \sim e^{-2k_I x}$

$$\nabla \cdot \mathbf{P} = -2k_I P_x = -2k_I v_{g,x} W = (\mathbf{E} \cdot \mathbf{J})_{disspation}$$

- Both spatial and temporal damping:

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{P} = -\left(2\omega_I + 2k_I v_{g,x}\right) W = (\mathbf{E} \cdot \mathbf{J})_{disspation}$$

Temporal and spatial damping

The theory for spatial and temporal damping can be derived from the dispersion equation, using $\omega \to \omega + i\omega_I$ and $\mathbf{k} \to \mathbf{k} + i\mathbf{k}_I$:

$$\det\left[\Lambda_{ij}^H(\omega + i\omega_I, \mathbf{k} + i\mathbf{k}_I) + K_{ij}^A(\omega + i\omega_I, \mathbf{k} + i\mathbf{k}_I)\right] = 0$$

Taylor expand for small ω_I, $\mathbf{k}_I$, and K_{ij}^A, like in Lec. 6:

$$\Lambda(\omega, \mathbf{k}) + \mathrm{i}\left(\omega_I \frac{\partial}{\partial\omega} + k_{I,i}\frac{\partial}{\partial k_i}\right)\Lambda(\omega, \mathbf{k}) + \lambda_{ij}(\omega, \mathbf{k})K_{ij}^A(\omega, \mathbf{k}) + \cdots = 0$$

where: $\Lambda(\omega, \mathbf{k}) \equiv \det\left[\Lambda_{ij}^H(\omega, \mathbf{k})\right]$.

1. Neglect small terms; approximate dispersion equation: $\Lambda(\omega, \mathbf{k}) = 0$
2. First order correction provides a relation between ω_I, $\mathbf{k}_I$, and K_{ij}^A:

$$\left(\omega_I \frac{\partial}{\partial\omega} + k_{I,i}\frac{\partial}{\partial k_i}\right)\Lambda(\omega, \mathbf{k}) = i\lambda_{ij}(\omega, \mathbf{k})K_{ij}^A(\omega, \mathbf{k})$$

$$= \left\{\text{from Lec 6: } \lambda_{ij} = \lambda_{ss} e_{M,i}^* e_{M,j}\right\} =$$

$$\Rightarrow \left(\omega_I \frac{\partial}{\partial\omega} + k_{I,i}\frac{\partial}{\partial k_i}\right)\Lambda(\omega, \mathbf{k}) = i\lambda_{ss}(\omega, \mathbf{k})K_M^A(\omega, \mathbf{k})$$

Here $K_M^A := e_{M,i}^* K_{ij}^A e_{M,j}$

Temporal and spatial damping

Repeat:

$$\omega_I \frac{\partial \Lambda(\omega, \mathbf{k})}{\partial \omega} + \mathbf{k}_I \cdot \frac{\partial \Lambda(\omega, \mathbf{k})}{\partial \mathbf{k}} = i\lambda_{ss}(\omega, \mathbf{k}) K_M^A(\omega, \mathbf{k})$$

Divide by $\partial \Lambda(\omega, \mathbf{k})/\partial \omega$:

$$\omega_I - \mathbf{v}_g \cdot \mathbf{k}_I = i \frac{\lambda_{ss}(\omega, \mathbf{k})}{\partial \Lambda(\omega, \mathbf{k})/\partial \omega} K_M^A(\omega, \mathbf{k})$$

Here $\mathbf{v}_g$ is the group velocity!

Note that:

$$\mathbf{v}_g = \frac{\partial \omega_M(\mathbf{k})}{\partial \mathbf{k}} = - \left. \frac{\partial \Lambda(\omega, \mathbf{k})/\partial \mathbf{k}}{\partial \Lambda(\omega, \mathbf{k})/\partial \omega} \right|_{\omega = \omega_M(\mathbf{k})}$$

Thus, the "*damping per meter*" and the "*damping per second*" are related by the group velocity – as if the wave energy was transported by the group velocity!

In homogeneous media, wave energy is transported with the group velocity

Ratio between electric and total energy density

For pure temporal damping, $\mathbf{k}_I = 0$, we get the damping rate $\gamma = -2\omega_I$:

$$\gamma = -2\omega_I = -i2\frac{\lambda_{ss}(\omega,\mathbf{k})}{\partial\Lambda(\omega,\mathbf{k})/\partial\omega} K_M^A(\omega,\mathbf{k})$$

Cmp. previous expression: $\gamma = -i2\omega_M(\mathbf{k})R_M(\mathbf{k})K_M^A(\omega,\mathbf{k})$

$$R_M(\mathbf{k}) = \left.\frac{\lambda_{ss}(\omega,\mathbf{k})}{\omega\partial\Lambda(\omega,\mathbf{k})/\partial\omega}\right|_{\omega=\omega_M(\mathbf{k})}$$

Remember, $R_M(\mathbf{k}) := \frac{W_M^E(\mathbf{k})}{W_M(\mathbf{k})}$

Since $W_{M,V}^E = \varepsilon_0\left|\omega_M a_{M,V}\right|^2/V$ and $W_M^M(\mathbf{k}) = n_M^2\left|1-|\boldsymbol{\kappa}\cdot\mathbf{e}_M(\mathbf{k})|\right|^2 W_M^E(\mathbf{k})$, the equation above allows us to calculate:

- Total energy: $W_M = W_M^E/R_M$
- Particle (kinetic) energy: $W_M^P = W_M - W_M^E - W_M^M$

$$W_M^P = W_M^E\left[\frac{1}{R_M(\mathbf{k})} - 1 - n_M^2\left|1-|\boldsymbol{\kappa}\cdot\mathbf{e}_M(\mathbf{k})|\right|^2\right]$$

Simplified version of R_M

To simplify $R_M(\mathbf{k})$ start from:

$$\frac{\partial \Lambda}{\partial \omega} = \{\text{from Lec.}6\} = \lambda_{ij}\frac{\partial \Lambda_{ij}}{\partial \omega} = \lambda_{ss}\frac{\partial}{\partial \omega} e_i^* \Lambda_{ij} e_j =$$

$$= \lambda_{ss}\frac{\partial}{\partial \omega} e_i^* e_j \left[\frac{c^2 k^2}{\omega^2}\left(\kappa_i \kappa_j - \delta_{ij}\right) + K_{ij}\right] =$$

$$= \lambda_{ss} e_i^* e_j \left[-2\frac{c^2 k^2}{\omega^3}\left(\kappa_i \kappa_j - \delta_{ij}\right) + \frac{\partial}{\partial \omega} K_{ij}\right] =$$

$$= \left\{\text{use wave eq.: } \Lambda_{ij} e_j = 0 \rightarrow c^2 k^2\left(\kappa_i \kappa_j - \delta_{ij}\right)/\omega^2 e_j = -K_{ij} e_j\right\} =$$

$$= \lambda_{ss} e_i^* e_j \left[\frac{2}{\omega} K_{ij} + \frac{\partial}{\partial \omega} K_{ij}\right] = \lambda_{ss}\frac{1}{\omega^2}\frac{\partial}{\partial \omega}\omega^2 e_i^* K_{ij} e_j = \lambda_{ss}\frac{1}{\omega^2}\frac{\partial}{\partial \omega}\omega^2 K_M$$

$$R_M(\mathbf{k}) = \left.\frac{\lambda_{ss}(\omega, \mathbf{k})}{\omega \partial \Lambda(\omega, \mathbf{k})/\partial \omega}\right|_{\omega = \omega_M(\mathbf{k})} = \left.\frac{\omega}{\partial \omega^2 K_M(\omega, \mathbf{k})/\partial \omega}\right|_{\omega = \omega_M(\mathbf{k})}$$

Note: No assumptions, we're only used the wave equation!

Simplified version of R_M

Example: Consider transverse wave in an electron gas: $K_M(\omega)^2 = 1 - \omega_p^2/\omega^2$.

$$\frac{\partial}{\partial\omega}\omega^2 K_M(\omega) = 2\omega \rightarrow R_M = \frac{1}{2}$$

The magnetic energy: $W_M^M = n_M^2 W_M^E = \left(1 - \omega_p^2/\omega^2\right) W_M^E$

Particle energy energy: $W_M^P = W_M - W_M^E - W_M^M = \omega_p^2/\omega^2 W_M^E$

There is also a further simplification when for non-spatially dispersive media ($K_{ij}(\omega)$ is independent of $\mathbf{k}$).

Study the $e_{M,i}^*$ component of the wave equation:

$$e_{M,i}^*(\mathbf{k})\Lambda_{ij}(\omega,\mathbf{k})e_{M,j}(\mathbf{k}) = 0$$
$$n^2(|\boldsymbol{\kappa}\cdot\mathbf{e}_M|^2 - 1) + e_{M,i}^*(\mathbf{k})K_{ij}(\omega,\mathbf{k})e_{M,j}(\mathbf{k}) = 0$$
$$K_M(\omega,\mathbf{k}) = n^2(1 - |\boldsymbol{\kappa}\cdot\mathbf{e}_M|^2)$$

$$R_M(\mathbf{k}) = \frac{1}{(1 - |\boldsymbol{\kappa}\cdot\mathbf{e}_M|^2)2n_M(\omega)\partial\omega n_M(\omega)/\partial\omega}$$

Energy continuity

An energy conservation equation can be derived from the 1st order term in the expanded dispersion equation:

$$\omega_I - \mathbf{k}_I \cdot \mathbf{v}_g = i \frac{\lambda_{ss}(\omega, \mathbf{k})}{\partial\Lambda(\omega, \mathbf{k})/\partial\omega} K_M^A(\omega, \mathbf{k})$$

Multiply with $-i2W_M$:

$$-i2\omega_I W_M + i2\mathbf{k}_I \cdot \mathbf{v}_g\, W_M = 2\omega_M R_M K_M^A W_M$$

$$\underbrace{i2\omega_I W_M}_{\sim \frac{\partial}{\partial t} W_M} - \underbrace{i2\mathbf{k}_I \cdot \mathbf{F}_M}_{\sim \nabla \cdot \mathbf{F}_M} = -\gamma W_M$$

Thus, this is a *continuity equation for wave energy*!

The energy density flux is: $\mathbf{F}_M = \mathbf{v}_g W_M$, thus energy is transported by $\mathbf{v}_g$!!

Particle contribution to the flux

The energy density flux, $\mathbf{F}_M = \mathbf{v}_g W_M$, includes both electromagnetic and kinetic parts: $\mathbf{F}_M = \mathbf{F}_M^{EM} + \mathbf{F}_M^{P}$

We know that the electromagnetic part is:

$$\mathbf{F}_M^{EM} = \frac{\langle \mathbf{E}\times\mathbf{B}\rangle}{\mu_0} = \frac{2c^2}{\omega_M(\mathbf{k})} Re\{\mathbf{k} - \mathbf{e}_M(\mathbf{e}_M^* \cdot \mathbf{k})\} W_M^E$$

Next we'll derive an expression for the kinetic flux, $\mathbf{F}_M^P$. Start from…

$$\mathbf{F}_M = \mathbf{v}_g W_M = -\frac{\partial\Lambda(\omega,\mathbf{k})/\partial\mathbf{k}}{\partial\Lambda(\omega,\mathbf{k})/\partial\omega} \frac{W_M^E}{R_M} =$$

$$\ldots = -\frac{\varepsilon_0 \omega^3 \left|a_{M,V}\right|^2}{V \lambda_{ss}} \frac{\partial}{\partial\mathbf{k}} \Lambda(\omega,\mathbf{k})\Big|_{\omega=\omega_M(\mathbf{k})}$$

Following Melrose & McPhedran, Eqs. 15.26-15.28:

Warning!
Error in Eq. (15.31)

$$\mathbf{F}_M^P = -\omega_M(\mathbf{k}) e_{M,i}^* e_{M,j} \frac{\partial K_{ij}}{\partial\mathbf{k}} W_M^E$$

***Note**: Only for spatially dispersive media is there a kinetic energy flux!*

Summary (1)

Continuity equation for wave energy:

$$\frac{\partial}{\partial t}(W^E + W^M + W^P) + \nabla\cdot(\mathbf{E}\times\mathbf{B}/\mu_0 + \mathbf{F}^P) = -(\mathbf{J}\cdot\mathbf{E})_{dissipative}$$

Terms specific to dispersive media

In dispersive media we need to work in Fourier space…but all terms are non-linear!!

To handle this we studied the volume and time average of each term.

- Infinite integrals that required careful evaluation using function truncated over a volume, V (notation: $K_M = e^*_{M,i}K_{ij}e_{M,j}$)
- Start from :

$$\mathbf{A}(\omega,k) = a_M(\mathbf{k})\mathbf{e}_M(\mathbf{k})2\pi\delta\big(\omega-\omega_M(\mathbf{k})\big) + a^*_M(-\mathbf{k})\,\mathbf{e}^*_M(-\mathbf{k})2\pi\delta(\omega+\omega_M(-\mathbf{k}))$$

$$\begin{cases} W^E_{M,V}(\mathbf{k}) = \dfrac{1}{V}\varepsilon_0\left|\omega_M(\mathbf{k})a_{M,V}(\mathbf{k})\right|^2 \\ W^M_{M,V}(\mathbf{k}) = n_M(\mathbf{k})^2\left|1-|\boldsymbol{\kappa}\cdot\mathbf{e}_M(\mathbf{k})|\right|^2 W^E_M(\mathbf{k}) \\ R_M(\mathbf{k}) \equiv \dfrac{W^E_{M,V}(\mathbf{k})}{W_{M,V}(\mathbf{k})} = \left.\dfrac{\omega}{\partial\omega^2 K_M(\omega,\mathbf{k})/\partial\omega}\right|_{\omega=\omega_M(\mathbf{k})} \end{cases}$$

$$\begin{cases} \mathbf{F}^P_{M,V}(\mathbf{k}) = -W^E_{M,V}(\mathbf{k})e^*_{M,i}e_{M,j}\omega_M\dfrac{\partial K_{ij}(\omega,\mathbf{k})}{\partial\mathbf{k}} \\ \gamma(\mathbf{k}) = -i2\omega_M(\mathbf{k})R_M(\mathbf{k})K^A_M(\omega_M(\mathbf{k}),\mathbf{k}) \end{cases}$$

Summary (2)

We have shown that for homogeneous media, the wave energy travels with the group velocity

$$\mathbf{v}_g = \frac{\partial \omega_M(\mathbf{k})}{\partial \mathbf{k}} = -\left.\frac{\partial \Lambda(\omega, \mathbf{k})/\partial \mathbf{k}}{\partial \Lambda(\omega, \mathbf{k})/\partial \omega}\right|_{\omega=\omega_M(\mathbf{k})}$$

In Fourier space there is an analogue to the continuity equation

$$\underbrace{i2\omega_I W_M(\mathbf{k})}_{\sim \frac{\partial}{\partial t} W_M} - \underbrace{i2\mathbf{k}_I \cdot \mathbf{F}_M(\mathbf{k})}_{\sim \nabla \cdot \mathbf{F}_M = \nabla \cdot (W_M \mathbf{v}_g)} = -\gamma(\mathbf{k}) W_M(\mathbf{k})$$