

Homework 5

Classification and Shock Tube

due February 25, 2016

Task 1 : Classification of partial differential equations

A simplified form of the one dimensional shallow water equations is considered. These partial differential equations are expressed in terms of velocity at the water column $u(x, t)$ and potential energy $\phi(x, t)$:

$$\begin{cases} u_t + \left(\frac{u^2}{2} + \phi\right)_x = 0 \\ \phi_t + (u\phi)_x = 0 \end{cases} \quad (x, t) \in \mathbb{R} \times [0, \infty). \quad (1)$$

The shallow water system can be written in conservative form as follows

$$U_t = F(U)_x = 0, \quad (2)$$

with the state variables and fluxes

$$U := \begin{bmatrix} u \\ \phi \end{bmatrix}, \quad F(U) := \begin{bmatrix} u^2/2 + \phi \\ u\phi \end{bmatrix}. \quad (3)$$

a) Rewrite Eq. (2) in the so called quasi-linear matrix form

$$U_t + AU_x = 0, \quad A(u, \phi) \in \mathbb{R}^{2 \times 2}. \quad (4)$$

b) Calculate the eigenvalues of A and classify the type of the PDE system.

c) Now, let

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}. \quad (5)$$

This matrix has a complete set of eigenvalues and eigenvectors. Diagonalize the matrix and find a characteristic form

$$W_t + \Lambda W_x = 0, \quad (6)$$

where W is a state vector including the characteristic variables and Λ a diagonal matrix containing the eigenvalues of the system. This equation has an exact solution. Find the solution analytically using the initial conditions

$$u = u_0, \quad \phi = \phi_0, \quad (x, t) \in \mathbb{R} \times \{0\}. \quad (7)$$

What is the expected behaviour of the solution?

Task 2 : Shock tube

In this task we consider the flow inside a shock tube. A shock tube is a pipe, closed at both ends, with a diaphragm in the middle separating a region with high-pressure gas and a region with low-pressure gas. The initial condition for the density, ρ , is given by (see also figure 1)

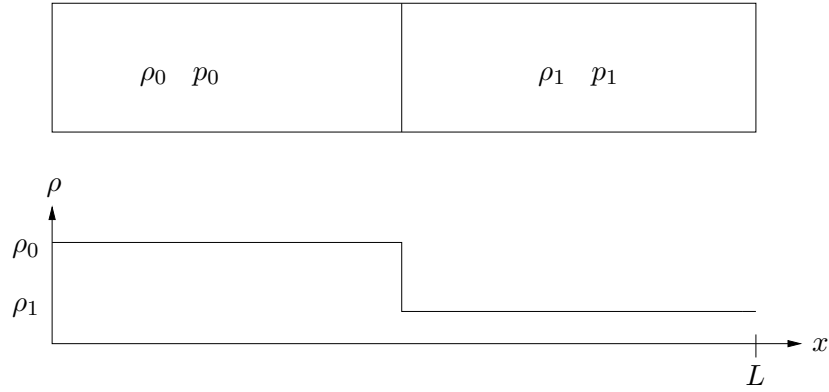


Figure 1: Initial condition.

$$\rho(x, 0) = \begin{cases} \rho_0, & \text{if } x \leq L/2, \\ \rho_1, & \text{if } x > L/2, \end{cases}$$

and the fluid is at rest, *i.e.* $u(x, 0) = 0$. L is the length of the tube.

The motion of a barotropic gas (pressure is only a function of the density) in the shock tube can be described by the 1D Euler equations

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= 0, \\ p &= K\rho^\gamma, \end{aligned} \quad (8)$$

where K is a constant determined by the initial conditions and the isentropic expansion factor is $\gamma = 1.4$. The system of equations (8) can be written in conservative form as

$$U_t + F(U)_x = 0, \quad (9)$$

where $U = (\rho, \rho u)$, and $F(U) = (\rho u, \rho u^2 + p)$ is the flux vector. This equation can be solved numerically using the MacCormack scheme:

$$\begin{aligned} U_j^* &= U_j^n - \lambda[F(U_{j+1}^n) - F(U_j^n)] && \text{Predictor step,} \\ U_j^{n+1} &= \frac{1}{2}(U_j^n + U_j^*) - \frac{\lambda}{2}[F(U_j^*) - F(U_{j-1}^*)] && \text{Corrector step,} \end{aligned} \quad (10)$$

where $\lambda = \Delta t/\Delta x$. Usually, the numerical solution of these equations will show unphysical oscillations (so-called *wiggles*). In order to damp these oscillations, an artificial-viscosity term can be added to the right-hand side of the equations. System (9) is thus modified as follows:

$$U_t + \tilde{F}(U)_x = 0, \quad \tilde{F}(U)_x := F(U)_x - (\hat{\nu}U_x)_x, \quad (11)$$

where $(\hat{\nu}U_x)_x$ is the artificial-viscosity term, and where \tilde{F} is called the “modified flux function”. The artificial viscosity $\hat{\nu}$ should be a small value of the order of the grid spacing Δx . When computing U^* in the predictor step, U_x should be approximated by a backward difference; when computing U^{n+1} in the corrector step, U_x should be approximated by a forward difference. Since we want to add numerical viscosity in regions with high gradients, we use a density switch model in which the density is used to localise the shock:

$$\hat{\nu} = \Delta x V_s [C_2 sw(\rho) + C_0]. \quad (12)$$

The multiplication by $\Delta x V_s$ is needed in order to obtain the correct physical unit for the viscous term. Here Δx is the mesh size, and the parameter V_s is related to the convection speed (characteristic speed) of the solution and is chosen as

$$V_s = \max_x \{|u + c|, |u - c|\},$$

where c is the speed of sound. The function $sw(\rho)$ is a density switch that is computed as follows

$$sw(\rho) = \left| \frac{\partial^2 \rho}{\partial x^2} \right| \frac{\Delta x^2}{\bar{\rho}} \approx 2 \frac{|\rho_{j-1} - 2\rho_j + \rho_{j+1}|}{|\rho_{j-1} + 2\rho_j + \rho_{j+1}|}.$$

The second-order derivative of ρ is approximated by a second-order central difference and $\bar{\rho}$ is the mean value of ρ computed at the j -th grid point as $\bar{\rho}_j = (\rho_{j+1} + 2\rho_j + \rho_{j-1})/4$. Where large gradients are present in the density field $sw(\rho)$ will be of order one, and, on the contrary, where the density field is smooth $sw(\rho)$ will be approximately zero. The parameter C_2 should be chosen in order to obtain sufficient viscosity to damp the oscillations, and C_0 is a “background” diffusion parameter and should also be chosen. Both C_0 and C_2 should be of order one or less. The optimal values of C_0 and C_2 are usually determined after some experimentation with different values (see task c) below).

Your task is to complete a MATLAB code which solves equation (11) using the MacCormack formulation (10).

The following files can be downloaded from the course home page:

```
shocktube.m
artificial_visc.m
dx.m
mac_cormack.m
boundary_cond.m
flux_function.m
```

1. The main program `shocktube.m` needs to be completed with the appropriate calculation of the time step. Additionally if any pre/postprocessing operations are needed they should be added here. Note the use of global variables.
2. The file `flux_function.m` defines the flux function. Here the flux function for the system of equations (9) must be coded. Also, two lines in `mac_cormack.m` need to be completed.
3. `boundary_cond.m` sets the boundary conditions, *i.e.* $(\rho)_1$, $(\rho)_n$, $(\rho u)_1$ and $(\rho u)_n$, (n is the number of grid points). To set the boundary conditions we use the physical condition that the tube is closed at both ends, so homogeneous Dirichlet conditions $u(0, t) = u(L, t) = 0$ are imposed. The other condition corresponds to a numerical boundary condition imposed by setting the value of ρ at the boundaries as a zeroth order extrapolation from the value of the density inside the tube. This is a simplification of the concept of Riemann invariants,

$$\rho_1 = \rho_2, \quad \rho_n = \rho_{n-1}.$$

4. Finally, the artificial-viscosity model is implemented in `artificial_visc.m`. In this file you need to add the definition of the speed of sound and V_s .

The length of the tube is set to $L = 3$. Note that the pressure p needs to be updated at every time step with the equation $p = K\rho^\gamma$.

The following points should be addressed in your report (including the completed MATLAB codes and the plots of your results):

- a) To better understand this scheme, write the MacCormack scheme for the advection equation

$$u_t + f_x = 0, \quad f = au.$$

You will notice that it coincides with another scheme. What is the name of this scheme (see lectures)?

- b) The CFL stability condition is guaranteed by choosing $\Delta t = C_N \Delta x / u_{max}$ with the Courant number $C_N < 1$. Here, u_{max} is the maximum absolute value of the characteristic speeds. From the quasi-linear form of the equations,

$$\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} u & \rho \\ K\gamma\rho^{\gamma-2} & u \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = 0,$$

show that the characteristic speeds are $u \pm c$ with $c^2 = K\gamma\rho^{\gamma-1} = \gamma p / \rho$ where c is the speed of sound. Use this to set the time step in the main program `shocktube.m`.

- c) Having completed the MATLAB files, run the code. The initial jump breaks up into a rarefaction wave moving left and a shock moving right. Measure the shock speed s (this can be done by considering the shock location at different times). Check its correctness by computing the analytical shock speed s from the Rankine-Hugoniot condition

$$s(\rho_L - \rho_R) = \rho_L u_L - \rho_R u_R,$$

where ρ_L and u_L are the computed states at the left-hand side of the shock, and ρ_R and u_R are the values on the right-hand side of the shock and thus given by the initial conditions.

- d) Run the code up to final time $t_f \approx 0.003$ (round it with the number of time steps) with different Courant numbers C_N and for the values of the artificial viscosity parameters C_2 and C_0 given below and discuss the solution. For the last case (optimal parameters) run the code up to $t_f \approx 0.01$, after the shock is reflected.

C_0	C_2
0.05	0.45
0.05	0.05
0.4	0.05
0.05	0.25