Lecture 10: Multi-User Detection, Frequency Hop Spread Spectrum, and Continuous Phase Modulation
Advanced Digital Communications (EQ2410)\(^1\)

M. Xiao
CommTh/EES/KTH

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Overview

Lecture 9
- Direct sequence spread spectrum
- CDMA

Lecture 10: Frequency hop spread spectrum and continuous phase modulation
- Multi-User Detection
- Asymptotic Efficiency
- Frequency Hop Spread Spectrum
- Coded Frequency Hop Multiple Access
- Continuous Phase Modulation
- Minimum Shift Keying
- Gaussian MSK
- Receiver Design
Multi-User Detection

- Synchronous CDMA, $N$-dimensional received vector
  \[ r = \sum_{k=1}^{K} A_k b_k s_k + W = U b + W = S A b + W = s_b + W \]

with
  - the AWGN $W \sim N(0, \sigma^2 I)$,
  - the symbol vector $b = (b_1, \ldots, b_K)^T$,
  - the matrix of channel gains $A = \text{diag}(A_k)$
  - the matrix of spreading sequences $S = (s_1, \ldots, s_K)$,
  - and $U = (A_1 s_1, \ldots, A_K s_K)$ and $s_b = U b$.

- Conventional detector/matched filtering can be expressed as
  \[ z_{MF}^H r = R A b + S A^H W \]

with the vector of MF decision statistics $z_{MF}$ and the correlation matrix $R = S^H S$.

Example: 2 users with spreading vectors $s_1$, $s_2$ and $\langle s_1, s_2 \rangle = \rho$

- Let
  \[ z_1 = \langle r, s_1 \rangle = A_1 b_1 + \rho A_2 b_2 + N_1 \]
  \[ z_2 = \langle r, s_2 \rangle = \rho A_1 b_1 + A_2 b_2 + N_2 \]

- $\langle r, s_b \rangle = A_1 b_1 z_1 + A_2 b_2 z_2$ with the matched-filter outputs $z_1, z_2$
- Decision regions for $A_1 = 1$, $A_2 = 2$ and $\rho = -0.5$

Notes
Multi-User Detection

- Linear zero-forcing (or decorrelating) detector for user $k$
  \[ z_{ZF}^k = \langle r, c_{ZF}^k \rangle = c_{ZF}^H r \quad \text{with} \quad c_{ZF}^k = (S^H S)^{-1} e_k = SR^{-1} e_k \]
- ZF detector expressed in matrix form for the vector
  \[ z_{ZF}^k = (Z_{ZF}^1, \ldots, Z_{ZF}^K)^T \]
  \[ z_{ZF}^k = c_{ZF}^H r = R^{-1} e^H r = R^{-1} z_{MF}. \]
- Linear MMSE detector for the $k$-th user by minimizing the
  \[ \text{MSE} = E[\| (r, c) - b_k \|^2] \]
  \[ z_{MMSE}^k = \langle r, c_{MMSE} \rangle = c_{MMSE}^H r \quad \text{with} \quad c_{MMSE} = E[rr^H]^{-1} \cdot E[b_k r] \]
- MMSE detector expressed in matrix form with
  \[ z_{MMSE} = (Z_{MMSE}^1, \ldots, Z_{MMSE}^K)^T \]
  \[ z_{MMSE} = A^H(SA)^{-1}E_{\text{I}}/E_{\text{S}} - 1 r \]
- Decision feedback methods: successive interference cancellation.

Asymptotic Efficiency

- Error probability for MUD $P_{e}$, error probability in the single-user case $P_{e,\text{su}}$.
- Asymptotic efficiency
  \[ \eta = \lim_{\sigma^2 \to 0} \frac{\log P_{e}}{\log P_{e,\text{su}}} \]
- Example:
  \[ P_{e} = e^{-a/\sigma^2} \quad \text{and} \quad P_{e,\text{su}} = e^{-b/\sigma^2} \]
  \[ \Rightarrow \eta = a/b \]
- Asymptotic efficiency for ML detection: $\eta_{ML} = 1 - |\rho|^2$
- Asymptotic efficiency for ZF/MMSE detection:
  \[ \eta_{ZF} = \frac{\|P_j s_j\|^2}{\|s_j\|^2} = \frac{\|P_j s_j\|^2}{\|s_j\|^2} \]
  \[ \quad \rightarrow \text{independent of interference amplitudes} \]
- Asymptotic efficiency for matched filter (MF): constant error probability (i.e., $a = 0$) as $A_i/A_l$ becomes large.
- Near/far resistance: minimum of the asymptotic efficiency
Frequency Hop Spread Spectrum

- Direct sequence spread spectrum
  - The bandwidth of a narrowband signal is spread to a large bandwidth.
  - All the bandwidth is used all the time.
- Frequency hop spread spectrum
  - Narrowband signaling with varying carrier frequency
  - Hopping pattern specifies how the carrier frequency is changed.
  → On average the whole bandwidth is used.
  - Slow FH: multiple symbols per hop; fast FH: one symbol per hop.
  - Frequency diversity (GSM standard)
  - Randomized multiple access (Bluetooth)
  - Phase synchronization is difficult
    → non-coherent or differential coherent modulation

Coded Frequency Hop Multiple Access

- $K$ users share $q$ frequencies; each user has its own hopping pattern; users are synchronized;
  - Probability that a collision occurs ("hit")
    \[ p_{\text{hit}} = 1 - \left( \frac{q-1}{q} \right)^{K-1} \]
  - Collision leads to an erasure of the symbols of that hop; $p_{\text{hit}}$ gives the erasure probability.
  - Coded system: use an erasure-correcting code across the hops.
  - Example: a $(n, k)$ Reed-Solomon code can correct $(n-k)$ erasures; probability that a frame gets lost:
    \[ P_F = \sum_{h=n-k+1}^{n} \binom{n}{h} p_{\text{hit}}^h (1 - p_{\text{hit}})^{n-h} \]
Continuous Phase Modulation

- Signal of the form \( e^{j\theta(t)} \), where \( \theta(t) \) is a continuous function of \( t \) carrying the data.
- Passband signal: \( u_p(t) = \cos(2\pi f_c t + \theta(t)) \)
- Constant envelope: signal can be recovered even if it passes through severe nonlinearities.
- Example: \( u_p(t) \) through a limiter function \( \text{sign}() \)
  \[
  v_p(t) = \text{sign}(u_p(t)) = \text{sign}(\cos(2\pi f_c t + \theta(t))) \\
  \approx a_1 \cos(2\pi f_c t + \theta(t)) + a_2 \cos(3(2\pi f_c t + \theta(t))) + \ldots
  
  \rightarrow u_p(t) \text{ can be recovered by low-pass filtering!}
  
  - Counterexample: QPSK with a non-ideal pulse is not CPM; it has as well the form \( e^{j\theta(t)} \) but the envelope can become zero, e.g., for transition from -1 to +1.

Minimum Shift Keying

- Minimum shift keying (MSK, both linear modulation and CPM)
- Transmission of 2 bits \( b_c[k], b_s[k] \in \{\pm1\} \)
  \[
  u(t) = \sum_k (b_c[k] \rho(t - 2kT_b) + j b_s[k] \rho(t - 2kT_b - T_b))
  
  \text{with } \rho(t) = \sin(\pi/T_s t) I_{[0,2T_b]} \text{ and } T_s = 2T_b
Minimum Shift Keying

- Alternative formulation (restricting to the interval \([0, 2T_b]\))

\[
0 \leq t \leq T_b : \quad u(t) = b_0[0] \sin \frac{\pi t}{2T_b} + j b_0[-1] \cos \frac{\pi t}{2T_b} = e^{j b_0[-1](\frac{\pi}{2} - b_0[0]\frac{\pi t}{2T_b})}
\]

with \(a[0] = -b_0[0]/b_0[-1] = -b_0[0] \cdot b_0[-1]\)

\[
T_b \leq t \leq 2T_b : \quad u(t) = b_0[0] \sin \frac{\pi t}{2T_b} - j b_0[1] \cos \frac{\pi t}{2T_b} = e^{j (a[1]\frac{\pi}{2} - \frac{\pi}{2} b_0[1])} = e^{j b_0[1](b_0[0]\frac{\pi t}{2T_b} - \frac{\pi}{2})}
\]

with \(a[1] = b_0[0]/b_0[1] = b_0[0] \cdot b_0[1]\).

- Observation: \(\theta(t)\) is piecewise linear

![Minimum Shift Keying Diagram](image)

Minimum Shift Keying

- With the phase pulse

\[
\phi(t) = \begin{cases} 
0 & \quad t < 0, \\
\frac{\pi t}{2T_b} & \quad 0 \leq t \leq T_b, \\
\frac{\pi}{2} & \quad t \geq T_b.
\end{cases}
\]

the phase signal is given as

\[
\theta(t) = \sum_n a[n]\phi(t - nT_b)
\]

- Instantaneous frequency

\[
f(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} = \sum_n a[n]g(t - nT_b)
\]

with the MSK frequency pulse

\[
g(t) = \frac{1}{4T_b}[0, T_b](t)
\]

→ instantaneous frequency in MSK is modulated by a rectangular pulse.
**Gaussian MSK**

- Special case of MSK: the frequency pulse $g(t)$ is smoothened by a Gaussian filter (impulse response $h(t)$, frequency response $H(f)$)

$$h(t) = \frac{1}{2\pi\nu^2} e^{-\frac{t^2}{2\nu^2}} \quad \text{and} \quad H(f) = \exp \left( -\left( \frac{f}{B} \right)^2 \log \frac{2}{2} \right)$$

with $\nu^2 = \ln 2/(4\pi^2B^2)$.

- G-MSK frequency pulse

$$g_{GMSK}(t) = \frac{1}{4T} \left[ Q\left( \frac{t - \tau}{\nu} \right) - Q\left( \frac{t}{\nu} \right) \right]$$

**Receiver Design**

- Suboptimal approach: since the frequency is modulated by the symbols $a[n]$, FM demodulation to get $f(t)$ and linear demodulation of $f(t)$. → poor performance for dispersive channels!

- Maximum likelihood: Viterbi on the phase trellis (there are only 4 distinct values possible at integer multiples of $T_b$ → high complexity for dispersive channels!

- Laurent approximation

  - CPM signals can be well approximated by linear modulation

$$u(t) = \exp \left( \sum_n a[n]\phi(t - nT) \right) \approx \sum_n B[n]s(t - nT)$$

  with pseudo symbols $B[n]$.

  → Modulation pulse is not a Nyquist pulse

- For transmission over ISI channels with $g_c(t)$, detection of $B[n]$ using channel equalization techniques (e.g., MLSE, linear ZF/MMSE, DFE).

  • See example in the book!