Lecture 11: Space-Time Communication / MIMO
Advanced Digital Communications (EQ2410)[1]

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10:00-12:00, B22

[1] Textbook: U. Madhow, *Fundamentals of Digital Communications*, 2008

Notes

Overview

1. Overview
2. Space-Time Channel Modeling
3. Information Theoretic Limits
4. Spatial Multiplexing
5. Space-Time Coding
6. Beamforming

Notes

Overview

Space-Time Communication

- Communication systems with multiple transmit and receive antennas
- Multiple-input/multiple-output systems (MIMO)
- Multiple transmit/receive antennas → diversity, increased capacity
- Multiple antennas in communication systems
  - Base stations: many antennas
  - Mobiles: 1-2 antennas
  - WIFI (IEEE 802.11n): 2-4 antennas

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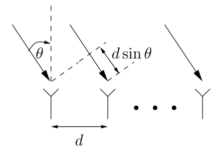
Space-Time Channel Modeling

- Linear antenna array, m antennas, spacing d



- Phase difference ϕ and delay τ between neighboring antennas

$$\phi(\theta) = \frac{2\pi d \sin\theta}{\lambda}, \quad \text{with } \lambda = c/f_c$$

$$\tau = \frac{d \sin\theta}{c}$$

- Received complex baseband signal at antenna i: $y(t - i\tau)e^{j(i-1)\phi}$ ($y(t)$: received signal at the first antenna)
- Narrowband assumption: bandwidth of $y \ll f_c \Rightarrow y(t - i\tau) \approx y(t)$
  - → Gain vector $\mathbf{a}(\theta) = (1, e^{j\phi}, \ldots, e^{j(m-1)\phi})^T$
  - → Array manifold: $\{\mathbf{a}(\theta)\}$, for $\theta \in [-\pi/2, \pi/2]$

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Space-Time Channel Modeling

- Multipath propagation (M paths, $T_m \ll T_s$):

$$\mathbf{h} = (h_1, \dots, h_m)^T = \sum_{i=1}^{M} g_i \mathbf{a}(\theta_i)$$

  - Gain of the i-th multipath component: g_i
  - Angle of the i-th multipath component: θ_i

→ Central limit theorem: $\mathbf{h}$ is zero-mean, proper complex Gaussian with covariance matrix

$$\mathbf{C}_h = \sum_{i=1}^{M} |g_i|^2 \mathbf{a}(\theta_i)\mathbf{a}(\theta_i)^H$$

- Power-angle profile: $P(\theta)$
  - Power density for a given angle θ ($\int P(\theta)d\theta = 1$)
  - For a large number of multipath components, we get

$$\mathbf{C}_h = \int \mathbf{a}(\theta)\mathbf{a}(\theta)^H P(\theta)d\theta$$

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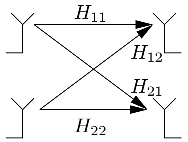
Space-Time Channel Modeling

- MIMO
  - N_T transmit antennas, transmit manifold $\mathbf{a}_T(\theta)$, departure angle θ
  - N_R receive antennas, receive manifold $\mathbf{a}_R(\gamma)$, arrival angle γ
- Channel characterization for narrowband signaling

$$\mathbf{H} = \sum_l g_l \mathbf{a}_R(\gamma_l)\mathbf{a}_T(\theta_l)^T$$

  - $\mathbf{H}$: $(N_R \times N_T)$ matrix
  - j-th column gives the receive array response to the j-th transmit antenna



$$\boldsymbol{H} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

  - Line-of-sight (LOS) link: $\mathbf{H} = \mathbf{a}_R(\gamma)\mathbf{a}_T(\theta)^T$
  - Rich scattering: H_{ij} i.i.d., zero-mean, proper complex Gaussian

Notes

Information Theoretic Limits
– Channel Capacity

- Channel model: $N_R \times N_T$ MIMO channel

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w} \quad \text{with} \quad \mathbf{w} \sim CN(\mathbf{0}, 2\sigma^2 \mathbf{I})$$

- Channel capacity (channel unknown at the transmitter)

$$C = \max_{\mathbf{C}_x} I(\mathbf{x}; \mathbf{y}) = \max_{\mathbf{C}_x} \log\det\left(\mathbf{I} + \frac{1}{2\sigma^2}\mathbf{H}\mathbf{C}_x\mathbf{H}^H\right)$$

with the transmit covariance matrix $\mathbf{C}_x$.

- Helpful tool: use a singular value decomposition to reduce the MIMO channel into a number of k parallel scalar channels.

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Information Theoretic Limits
– Singular value decomposition

- Define

$$\mathbf{W} = \begin{cases} \mathbf{H}^H\mathbf{H} & \text{if } N_T \le N_R \\ \mathbf{H}\mathbf{H}^H & \text{if } N_R \le N_T \end{cases} \quad \text{and} \quad \begin{aligned} m &= \min(N_T, N_R) \\ M &= \max(N_T, N_R) \end{aligned}$$

$\mathbf{W}$ is nonnegative definite (i.e., all eigenvalues are nonnegative), and $\mathbf{W}$ has dimension $(m \times m)$.

- Consider the case $N_T \le N_R$; i.e., $m = N_T$ and $M = N_R$.
- Let $\mathbf{v}_i$ be the length-m eigenvector to the eigenvalue $\lambda_i \ge 0$ of $\mathbf{W}$ (i.e., $\mathbf{W}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for $\lambda_i > 0$ and $\mathbf{W}\mathbf{v}_i = \mathbf{0}$ for $\lambda_i = 0$).
- Matrix of eigenvectors $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_m)$, normalization $\mathbf{V}^H\mathbf{V} = \mathbf{I}$ $\rightarrow$ eigenvectors are an orthonormal basis of the input space.
- Assumption: k nonzero eigenvalues $\lambda_i > 0$ for $i \in \{1, \dots, k\}$, and $m - k$ eigenvalues $\lambda_i = 0$ for $i \in \{k+1, \dots, m\}$.

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Information Theoretic Limits
– Singular value decomposition

- Define the length-M (receive) vectors $\mathbf{u}_i = \lambda_i^{-\frac{1}{2}}\mathbf{H}\mathbf{v}_i$, $i \in \{1, \ldots, k\}$ (i.e., vectors $\mathbf{u}_i$ are orthonormal, $\mathbf{u}_i^H\mathbf{u}_j = \delta_{ij}$).
- Define $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}'_1, \ldots, \mathbf{u}'_{m-k})$ with $\mathbf{u}'_i$ such that $\mathbf{U}^H\mathbf{U} = \mathbf{I}_m$.
  $\rightarrow$ the vectors in $\mathbf{U}$ are an orthonormal basis for the output space.
- We can show that

$$\mathbf{H}\mathbf{x} = \sum_{i=1}^{m} \mathbf{u}_i\sqrt{\lambda_i}\mathbf{v}_i^H\mathbf{x} = \mathbf{U}\mathbf{D}\mathbf{V}^H\mathbf{x}$$

$$\mathbf{H} = \mathbf{U}\mathbf{D}\mathbf{V}^H$$

  with $\mathbf{D} = \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_m})$ $\rightarrow$ singular values $\sqrt{\lambda_i}$
- Proof: Use $\mathbf{x} = \sum_{i=1}^{m} \mathbf{v}_i\mathbf{v}_i^H\mathbf{x}$ and solve $\mathbf{H}\mathbf{x}$.
- If we now choose $\hat{\mathbf{y}} = \mathbf{U}^H\mathbf{y}$ and $\hat{\mathbf{x}} = \mathbf{V}^H\mathbf{x}$, we get

$$\hat{\mathbf{y}} = \mathbf{D}\hat{\mathbf{x}} + \hat{\mathbf{w}}$$

  $\rightarrow$ the channel is decomposed into k parallel channels!

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Information Theoretic Limits
– MIMO Capacity

- Channel known at the transmitter: use $\hat{\mathbf{y}} = \mathbf{D}\hat{\mathbf{x}} + \hat{\mathbf{w}}$ and do waterfilling for the symbols $\hat{\mathbf{x}}$:

$$C_{CSI-T} = \sum_{i=1}^{k} \log(1 + \frac{\lambda_i P_i}{2\sigma^2}) \quad \text{with} \quad P_i = \left[a - \frac{2\sigma^2}{\lambda_i}\right]^+$$

  and a such that $\mathrm{E}[\|\mathbf{x}\|^2] \leq P$ is fulfilled ($P_i = \mathrm{E}[|\hat{x}_i|^2]$).
- No channel-state information at the transmitter
  - Choose the input distribution $P(\mathbf{X})$ to maximize the mutual information

$$I(\mathbf{Y};\mathbf{X}) = H(\mathbf{Y}) - H(\mathbf{Y}|\mathbf{X})$$

  $\rightarrow$ Since $H(\mathbf{Y}|\mathbf{X}) = H(\mathbf{W})$ (i.e., the conditional entropy is independent of the input), it is sufficient maximize $H(\mathbf{Y})$.

  $\rightarrow$ $H(\mathbf{Y})$ is maximized if $\mathbf{Y}$ is proper complex Gaussian.
- (Differential) entropy for a complex Gaussian vector $\mathbf{Z} \sim CN(\mathbf{m}, \mathbf{C})$

$$H(\mathbf{Z}) = \log\det(\pi e\mathbf{C}) = \sum_{i=1}^{n}(\log(\lambda_i) + \log(\pi e))$$

  with λ_i denoting the eigenvalues of $\mathbf{C}$

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Information Theoretic Limits
– MIMO Capacity

- With $\mathbf{C}_w = 2\sigma^2\mathbf{I}$ and $\mathbf{C}_\mathbf{y} = \mathbf{H}\mathbf{C}_x\mathbf{H}^H + 2\sigma^2\mathbf{I}$ we get

$$I(\mathbf{Y};\mathbf{X}) = H(\mathbf{Y}) - H(\mathbf{W}) = \log\det(\mathbf{I} + \frac{1}{2\sigma^2}\mathbf{H}\mathbf{C}_x\mathbf{H}^H)$$

- Mutual information for spatially white input with $\mathbf{C}_x = P\mathbf{I}/N_T$

$$C_{white} = \log\det(\mathbf{I} + \frac{SNR}{N_T}\mathbf{H}\mathbf{H}^H) = \sum_{i=1}^{k}\log(1 + \frac{SNR}{N_T}\lambda_i)$$

with $SNR = P/2\sigma^2$.

→ Optimal for rich scattering when entries in $\mathbf{H}$ are i.i.d., zero-mean complex Gaussian

- Ergodic capacity for rich scattering

$$C_{rich\ scattering} = \sum_{i=1}^{k} E[\log(1 + \frac{SNR}{N_T}\lambda_i)]$$

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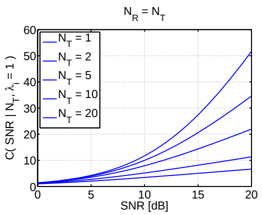
Spatial Multiplexing

- Motivation: MIMO capacity scales with $\min(N_T, N_R)$

$$C = \min(N_T, N_R)\mathrm{E}\left[\log\left(1 + \frac{SNR}{N_T}\lambda\right)\right]$$

→ How can this be achieved?



- Observation: MIMO system can be interpreted as a CDMA system

$$\mathbf{y} = b_1\mathbf{h}_1 + \ldots + b_{N_T}\mathbf{h}_{N_T} + \mathbf{w}$$

  - Each user/each stream is mapped to one antenna (or a group of antennas) → *spatial multiplexing*.
  - "Spreading codes" $\mathbf{h}_i$ are generated "by nature".
  - BLAST: Bell-Labs layered space-time architecture
  - Linear receiver processing: MMSE or ZF
  - Diversity: coding across several antennas.

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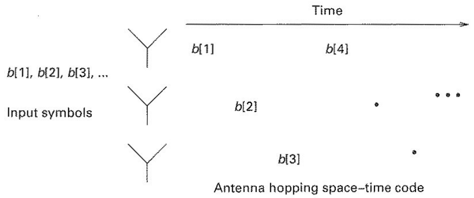
Space-Time Coding

- Motivation
  - Time diversity can only be achieved for (very) mobile users or for a quickly changing environment.
  - For slow or static users (e.g., often in WIFI) frequency and spatial diversity is important.
  - → Space-time codes for exploiting spatial diversity.
- Model
  - Narrowband, time-invariant system without time and frequency diversity
  - N_T transmit antennas, $N_R = 1$ receive antennas
    ($N_R > 1$: maximum ratio combining for antenna outputs)
    $$y[m] = \mathbf{h}\mathbf{x}[m] + w[m] = h_1 x_1[m] + \ldots + h_{N_T} x_{N_T}[m] + w[m]$$
    with $\mathbf{h} = (h_1, \ldots, h_{N_T})$.
- Capacity: $C(\mathbf{h}) = \log(1 + G \cdot SNR)$ with $G = \frac{\|\mathbf{h}\|^2}{N_T} = \frac{1}{N_T} \sum_{l=1}^{N_T} |h_l|^2$
  → fluctuations in h_i are averaged out; improved outage capacity.
- Problem: without structure in $\mathbf{x}$ the detection complexity is high (e.g., exponential in N_T for ML). → Space-time code design!

Notes

Space-Time Coding
– Antenna Hopping

- Use one transmit antenna at a time in an alternating fashion



[U. Madhow, *Fundamentals of Dig. Comm.*, 2008]

- Achievable rate: average of the rates of the sub-channels

$$C_{hopping}(\mathbf{h}) = \frac{1}{N_T} \sum_{l=1}^{N_T} \log(1 + |h_l|^2 SNR) < C(\mathbf{h})$$

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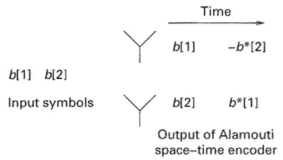
Space-Time Coding
– Alamouti Code

- Space-time code which achieves the capacity $C(\mathbf{h})$ for $N_T = 2$
- Two symbols $b[1]$ and $b[2]$ are transmitted using two channel uses

$$\mathbf{x}[1] = \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix} = \begin{bmatrix} b[1] \\ b[2] \end{bmatrix}$$
$$\mathbf{x}[2] = \begin{bmatrix} x_1[2] \\ x_2[2] \end{bmatrix} = \begin{bmatrix} -b^*[2] \\ b^*[1] \end{bmatrix}$$



[U. Madhow, *Fundamentals of Dig. Comm.*, 2008]

- The corresponding received signals

$$y[1] = h_1x_1[1] + h_2x_2[1] + w[1] = h_1b[1] + h_2b[2] + w[1]$$

and

$$y[2] = h_1x_1[2] + h_2x_2[2] + w[2] = -h_1b^*[2] + h_2b^*[1] + w[2]$$

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Space-Time Coding
– Alamouti Code

- Consider now

$$\mathbf{y} = \begin{bmatrix} y[1] \\ y^*[2] \end{bmatrix} = \underbrace{\begin{bmatrix} h_1 \\ h_2^* \end{bmatrix}}_{=\mathbf{v}_0} b[1] + \underbrace{\begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix}}_{=\mathbf{v}_1} b[2] + \begin{bmatrix} w_1 \\ w_2^* \end{bmatrix}$$

- Correlate $\mathbf{y}$ with $\mathbf{v}_0$ and $\mathbf{v}_1$ to get the decision variables

$$Z_1 = \mathbf{v}_0^H\mathbf{y}, \quad \text{for} \quad b[1],$$
$$Z_2 = \mathbf{v}_1^H\mathbf{y}, \quad \text{for} \quad b[2]$$

- Observation: $\mathbf{v}_0^H\mathbf{v}_1 = \mathbf{v}_1^H\mathbf{v}_0 = 0$, i.e., two parallel AWGN channels with the effective SNR $\|\mathbf{v}_0\|^2/2\sigma^2 = \|\mathbf{v}_1\|^2/2\sigma^2 = \|\mathbf{h}\|^2/2\sigma^2$.
- With $P_1 = P_2 = P/2$ the Alamouti code achieves the capacity $C(\mathbf{h})$.
- Optimal strategy if the transmitter does not know the channel $\mathbf{h}$.

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Beamforming

- If the channel vector $\mathbf{h}$ is known at the transmitter, we can use the spatial matched filter at the transmitter $\mathbf{h}^H$ by transmitting

$$\mathbf{x}[n] = \mathbf{h}^H b[n]$$

- Received signal at time n

$$y[n] = b[n] \cdot \sum_{k=1}^{N_T} |h_k|^2 + w[n] = b[n] \cdot \|\mathbf{h}\|^2 + w[n]$$

- Optimal strategy if the transmitter knows the channel $\mathbf{h}$.

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