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SF2705 Fourier Analysis
Fourier series and harmonic functions in the unit disk

We denote by γ a complex variable in the open or closed unit disk and we represent it in polar coordinates as $\gamma = re^{2\pi i x}$, $0 \le r \le 1$, $x \in \mathbb{R}$ as well as in the Cartesian coordinates as $\gamma = \xi + i\eta$.

A function u defined in the open unit disk is called a *harmonic* function if it satisfies the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0.$$

It is a standard fact from complex analysis that any real-valued harmonic function is a real part of an analytic function, i.e. $u(\gamma) = \operatorname{Re} f(\gamma)$, where f is analytic in the unit disk. Decomposing f into a Taylor series

$$f(\gamma) = \sum_{n=0}^{\infty} \hat{f}(n)\gamma^n,$$

we obtain

$$u(\gamma) = \hat{f}(0) + \frac{1}{2}\sum_{n\ge 1} \hat{f}(n)\gamma^n + \frac{1}{2}\sum_{n\le -1} \overline{\hat{f}(|n|)}\bar{\gamma}^{|n|},$$

where both series converge inside the unit disk. This means that any harmonic function is an infinite linear combination of functions γ^n, $n \ge 0$ and $\bar{\gamma}^{|n|}$, $n \le -1$. Obviously, the last conclusion remains true even for complex-valued harmonic functions. The converse is also true: any infinite linear combination, convergent in the unit disk, of functions γ^n, $n \ge 0$ and $\bar{\gamma}^{|n|}$, $n \le -1$ is a harmonic function.

Given a function g defined and continuous on the unit circle $\mathbb{T} = \{\gamma : |\gamma| = 1\}$, the Dirichlet problem for the Laplace equation consists in finding a function u defined and continuous in the closed unit disk, harmonic in the open disk and coinciding with g on the unit circle. If we search such a function u in the form

$$u(\gamma) = \sum_{n\ge 0} c_n\gamma^n + \sum_{n\le -1} c_n\bar{\gamma}^{|n|},$$

then we obtain a harmonic function (provided that the series converges in the disk which is true e.g. if coefficients c_n are bounded). Substituting (a bit formally first) boundary points $\gamma = e^{2\pi i x}$, we obtain a requirement

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x} = g(e^{2\pi i x})$$

which means that c_n are to be choosen as Fourier coefficients $\hat{g}(n)$ of the function $g(e^{2\pi i x})$. We obtain therefore formula

$$u(\gamma) = \sum_{n\ge 0} \hat{g}(n)\gamma^n + \sum_{n\le -1} \hat{g}(n)\bar{\gamma}^{|n|},\ |\gamma| < 1$$

and it remains to prove that the obtained function u can be extended continuously to the boundary of the disk by the function g.

Considering functions $u_r(e^{2\pi ix}) = u(re^{2\pi ix})$, we see that

$$u_r(e^{2\pi ix}) = \sum_{n=-\infty}^{\infty} \hat{g}(n) r^{|n|} e^{2\pi inx}.$$

This means that the function u_r is a convolution of g with the *Poisson kernel* P_r, where

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{2\pi inx} = \frac{1-r^2}{1-2r\cos(2\pi x)+r^2}$$

(see Exercise 1.4.8) By the result of Exercise 1.4.9, functions u_r converge uniformly to the function g, which gives the result.

The above arguments give also *the Poisson integral formula* for the solution u of the Dirichlet problem:

$$u(re^{2\pi ix}) = \int_0^1 \frac{1-r^2}{1-2r\cos(2\pi(x-y))+r^2}\, g(e^{2\pi iy})\, dy.$$

Another important relation between Fourier series and harmonic functions appear if we analyse harmonic conjugate functions. Remind that given a real-valued function u harmonic in the unit disk, another real-valued harmonic function v is called harmonic conjugate to u if the sum $u+iv$ is an analytic function. A standard theorem in complex analysis guarantees that such a harmonic conjugate function always exists at least locally (and hence globally in the whole disk since it is a simply connected domain) and it is unique up to an additive constant. Let us choose this constant so that the function v satisfies $v(0) = 0$. Now, we assume that the function u is represented as a series

$$u(\gamma) = \sum_{n\geq 0} \hat{u}(n)\gamma^n + \sum_{n\leq -1} \hat{u}(n)\bar{\gamma}^{|n|}.$$

A direct inspection shows then that one can choose the harmonic conjugate function v as

$$v(\gamma) = -i\left(\sum_{n\geq 1} \hat{u}(n)\gamma^n - \sum_{n\leq -1} \hat{u}(n)\bar{\gamma}^{|n|}\right).$$

If we look only at the boundary values of functions u and v, then we see that the mapping of u to v transforms the Fourier series $\sum_{n=-\infty}^{\infty} \hat{u}(n)e_n$ to the series $\sum_{n=-\infty}^{\infty} -i\,\mathrm{sign}\,(n)\hat{u}(n)e_n$, where sign is the signum function

$$\mathrm{sign}\,(n) = \begin{cases} 1, & n \geq 1; \\ 0, & n = 0; \\ -1, & n \leq -1. \end{cases}$$

This mapping is called *the Hilbert transform*. Obviously, it is a bounded operator in the space $L^2(S^1)$. But unfortunately, the Hilbert transform does not map the space $C(S^1)$ into itself, neither space $L^1(S^1)$ into itself (but it maps any space $L^p(S^1)$ into itself for all finite $p > 1$, this is the theorem of Marcel Riesz).