22.M/M/10 loss system with $\lambda = N/2\, s^{-1}$ and $E[X] = 1\, s$ (Erlang system).

(a) The conditions for Erlang's loss formula are fulfilled which means that P(call blocked)=$E_{10}(\rho_o)$ where ρ_o is the offered load.

$$\rho_o = \lambda \cdot E[X] = \frac{N}{2}$$ which means that P(call blocked)=$E_{10}(N/2) < 0,01$

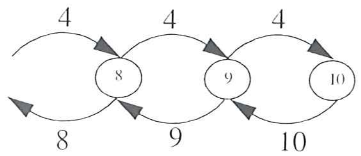
By using the Erlang tables we find that $\frac{N}{2} \le 4,4 \Rightarrow N \le 8,8$ to achieve a low enough blocking probability.
Therefore, maximum 8 sources can send jobs to the system.

(b) If N=8 then ρ_o=4.
Carried load=

$$\rho_c = \lambda_{eff} \cdot E[X] = \lambda \cdot (1 - E_{10}(\rho_o)) \cdot E[X] = \rho_o \cdot (1 - E_{10}(\rho_o)) \approx 3,98$$

Carried load per server $= \frac{\rho_o}{10} \approx 0,4$ Erlang, since the load is evenly distributed.

(c) The end of the Markov chain:



P(at least 8 servers busy) $= p_8 + p_9 + p_{10}$. Also $p_{10} = E_{10}(4) \approx 0,005308$
The "cut method" gives the following equations:

$$\begin{cases} 4p_9 = 10p_{10} \Rightarrow p_9 = \frac{5}{2} \cdot p_{10} \approx 0,01327 \\ 4p_8 = 9p_9 \Rightarrow p_8 = \frac{9}{4} \cdot p_9 \approx 0,02986 \end{cases}$$

Therefore, P(at least 8 servers busy)=0.0484

Solutions for the exercises in Chapter 5
(M/G/1-systems)

1. The service times in an M/M/1-system are exponentially distributed with mean $1/\mu$. This means that the density function for the service times, $b(t) = \mu e^{-\mu t}, t \geq 0$.

   Pollaczek-Khintcin's formula gives $\bar{N} = \rho + \dfrac{\lambda^2 E[X^2]}{2(1-\rho)}$, where $\rho = \lambda E[X] = \lambda/\mu$.

   Alternative 1:

   The second moment of the service times, $E[X^2]$, is calculated as:

   $$E[X^2] = \int_{-\infty}^{\infty} t^2 b(t)dt = \int_0^{\infty} t^2 \mu e^{-\mu t} dt = [-t^2 e^{-\mu t}]_0^{\infty} + \int_0^{\infty} 2te^{-\mu t} dt = \ldots = \frac{2}{\mu^2} = 2 \cdot E[X]^2$$

   Alternative 2:

   An alternative way to calculate the second moment is to use the Laplace tranform of $b(t)$.

   $$B^*(s) = \frac{\mu}{\mu + s}$$

   $$E[X^2] = \lim_{s \to 0} \frac{d^2}{ds} B^*(s) = \ldots = \lim_{s \to 0} \frac{2\mu}{(\mu + s)^3} = \frac{2}{\mu^2}$$

   For an M/M/1-system this means that: $\bar{N} = \rho + \dfrac{\lambda^2 \cdot 2/\mu^2}{2(1-\rho)} = \rho + \dfrac{\rho^2}{(1-\rho)} = \ldots = \dfrac{\rho}{1-\rho}$

2. Pollaczek-Khintcin's formula can be written as: $\bar{N} = \rho + \dfrac{\rho^2(1 + C^2)}{2(1-\rho)}$, where $\rho = \lambda E[X]$ and C^2 is the squared coefficent of variance. As can be seen from the formula, a larger value of C^2 means a higher mean number of jobs in the system for a specific arrival rate, if the systems have the same mean service time.

   M/M/1:

   $E[X]=1$, $C^2=1$ (see, Exercise 3 in Chapter 7)

   M/D/1:

   $E[X]=1$. The variance is zero for a deterministic service time, which means that

   $$C^2 = \frac{V[X]}{E[X^2]} = 0.$$

   M/E$_2$/1:

   $E[X]=1$. C^2 can be found by using Exercise 3 in Chapter 7: $C^2 = \dfrac{1}{r} = \dfrac{1}{2}$.

   M/H$_2$/1:

   The mean service time can be found by using the formula on page 82. Result: $E[X]=1$.

   There, you can also find the formula for C^2. Either calculate it explicitly, or just conclude that $C^2>1$.

   Result:

   The curves belong to the following systems (from lowest to highest curve): *M/D/1, M/E$_2$/1, M/M/1, M/H$_2$/1.*

3. Of course, it must be assumed that the system can be modelled as an M/G/1-system.

Combination of PK-formula and Little's theorem gives: $T = E[X] + \frac{\lambda E[X^2]}{2(1-\rho)}$,

where $\rho = \lambda E[X]$.

(a) Exponential distribution: $E[X^2] = \frac{2}{\mu^2} = 2 \cdot \left(\frac{1}{\mu}\right)^2 = 18$ (see, Exercise 1)

This means that $T = 3 + \frac{0.1 \cdot 18}{2(1-0.1 \cdot 3)} \approx 4.3$ seconds.

(b) Deterministic distribution: $V[X] = 0 \Rightarrow E[X^2] - E[X]^2 = 0 \Rightarrow E[X^2] = E[X]^2 = 9$.

This means that $T = 3 + \frac{0.1 \cdot 9}{2(1-0.1 \cdot 3)} \approx 3.6$ seconds.

(c) Density function for the service time:
$b(t) = 0.25 \cdot \delta(t-1) + 0.5 \cdot \delta(t-3) + 0.25 \cdot \delta(t-5)$, where $\delta(t)$ is the Dirac function.
This means that the Laplace transform for the service time is:

$$B^*(s) = 0.25 \cdot e^{-s} + 0.5 \cdot e^{-3s} + 0.25 \cdot e^{-5s}$$

$$E[X] = -\lim_{s \to 0} \frac{d}{ds} B^*(s) = 0.25 \cdot 1 + 0.5 \cdot 3 + 0.25 \cdot 5 = 3 \text{ seconds.}$$

$$E[X^2] = \lim_{s \to 0} \frac{d^2}{ds} B^*(s) = \lim_{s \to 0} (0.25 \cdot e^{-s} + 4.5 \cdot e^{-3s} + 6.25 \cdot e^{-5s}) = 11,$$

which means that $T = 3 + \frac{0.1 \cdot 11}{2(1-0.1 \cdot 3)} \approx 3.8$ seconds.

4. Combination of PK-formula and Little's theorem give: $W = \frac{\lambda E[X^2]}{2(1-\rho)}$ where $\rho = \lambda E[X]$.

(a) Exponential distribution: $E[X^2] = 2 \cdot E[X]^2$ (see, Exercise 1),

which means that $W = \frac{\lambda \cdot 2 \cdot E[X]^2}{2(1-\rho)} = \frac{\rho \cdot E[X]}{1-\rho}$.

(b) Deterministic distribution: $E[X^2] = E[X]^2$ (see, Exercise 3),

which means that $W = \frac{\lambda \cdot E[X]^2}{2(1-\rho)} = \frac{1}{2} \cdot \frac{\rho \cdot E[X]}{1-\rho}$.

(c) The average waiting time in the M/M/1-system is twice the average waiting time in the M/D/1-system.

5. In this exercise, we have to assume that packets arrive according to a Poisson process with mean 1500 packets per minute, which is equal to 25 packets per second. Then, if we assume that the queue is infinite, we can model the system as an M/G/1-system. The capactity of the server is 50kbit/sec. The variables that are searched for are $\bar{N}, \bar{N}_q, T$.

   (a) If the packets have an exponentially distributed length with mean 1000 bits, the service time for a packet will also be exponentially distributed with mean $E[X] = \frac{1000}{50000} = 0.02$ seconds.
   This means that we have an M/M/1-system, with λ=25 and $E[X]$=0.02.
   Let $\rho = \lambda E[X] = 0.5$.
   This means that:

$$\bar{N} = \frac{\rho}{1-\rho} = 1$$

$$\bar{N}_q = \bar{N} - \bar{N}_s = \frac{\rho}{1-\rho} - \rho = 0.5$$

   Use Little's theorem: $T = \frac{\bar{N}}{\lambda_{eff}} = \frac{\bar{N}}{\lambda} = 0.04$ seconds

   (b) In this case, 10% of the packets are 100 bits and 90% of the packets are 1500 bits.
   This means that the denstity function for the service time is

$$b(t) = 0.1 \cdot \delta\left(t - \frac{100}{50000}\right) + 0.9 \cdot \delta\left(t - \frac{1500}{50000}\right) = 0.1\delta(t-0.002) + 0.9\delta(t-0.03)$$

   where $\delta(t)$ is the Dirac function.
   The Laplace transform for $b(t)$ is $B^*(s) = 0.1e^{-0.002s} + 0.9e^{-0.03s}$.
   This means that:

$$E[X] = -\lim_{s \to 0} \frac{d}{ds} B^*(s) = 0.1 \cdot 0.002 + 0.9 \cdot 0.03 = 0.0272 \text{ seconds}$$

$$E[X^2] = \lim_{s \to 0} \frac{d^2}{ds} B^*(s) = 0.1 \cdot (0.002)^2 + 0.9 \cdot (0.03)^2 \approx 8.1 \cdot 10^{-4}$$

   Let $\rho = \lambda E[X] = 0.68$.
   Use the PK-formula:

$$\bar{N} = \rho + \frac{\lambda^2 E[X^2]}{2(1-\rho)} = \ldots \approx 1.47$$

$$\bar{N}_s = \rho = 0.68$$

$$\bar{N}_q = \bar{N} - \bar{N}_s = 0.79$$

   Use Little's theorem: $T = \frac{\bar{N}}{\lambda_{eff}} = \frac{\bar{N}}{\lambda} \approx 0.059$ seconds

6. We have an M/G/1-system with λ=10 calls per hour, which is equal to 1/6 calls per minute.
Before course:
M/M/1-system with $E[X]$=5 minutes.
Let $\rho = \lambda E[X] = 5/6$

$$\bar{N} = \frac{\rho}{1-\rho} = 5$$

In the following calculations we will also need the variance of the service time for an M/M/1-system: $V[X] = E[X^2] - E[X]^2 = E[X]^2$ (see, Exercise 1)
For this system this means that: $V[X] = 25$
Standard deviation: $\sigma = \sqrt{V[X]} = 5$
After course:
Mean service time has increased 10%: $E[X] = 1.1 \cdot 5 = 5.5$ minutes
Standard deviation has improved (decreased) 20%: $\sigma = 0.8 \cdot 5 = 4$
which means that

$$V[X] = \sigma^2 = 16$$

$$E[X^2] = V[X] + E[X]^2 = 46.25$$

Let $\rho = \lambda E[X] = 0.9167$

PK-formula: $\bar{N} = \rho + \dfrac{\lambda^2 E[X^2]}{2(1-\rho)} = \ldots \approx 8.62$

This means that the general performance of the system has decreased (is worse than before).

7. For an M/G/1-system, the generating function for the stationary probability distribution is given by $P(z) = \dfrac{B^*(\lambda(1-z))(1-z)(1-\rho)}{B^*(\lambda(1-z))-z}$ where $B^*(\lambda(1-z))$ is the Laplace transform of the service time distribution, $B^*(s)$, evaluated in $s=\lambda(1-z)$ and $\rho = \lambda E[X]$.

   (a) M/M/1-system:

   The density function for the service time is given by $b(t) = \mu e^{-\mu t}$ where $E[X] = 1/\mu$.

   Therefore, the Laplace transform is given by $B^*(s) = \dfrac{\mu}{\mu + s}$, and evaluated in $s=\lambda(1-z)$ this gives $B^*(\lambda(1-z)) = \dfrac{\mu}{\mu+\lambda(1-z)}$.

   $$\text{Therefore, } P(z) = \frac{\dfrac{\mu(1-z)(1-\rho)}{\mu+\lambda(1-z)}}{\dfrac{\mu}{\mu+\lambda(1-z)} - z} = \frac{\mu(1-z)(1-\rho)}{\mu(1-z)-\lambda z(1-z)} = \frac{\mu(1-\rho)}{\mu - \lambda z} = \frac{1-\rho}{1-\rho z}$$

   (b) M/D/1-system:

   The density function for the service time is given by $b(t) = \delta(t-a)$ where $a = E[X]$, which means that $B^*(s) = e^{-as} \Rightarrow B^*(\lambda(1-z)) = e^{-a\lambda(1-z)}$.

   $$\text{Therefore, } P(z) = \frac{e^{-a\lambda(1-z)}(1-z)(1-\rho)}{e^{-a\lambda(1-z)} - z} = \frac{(1-z)(1-\rho)}{1-ze^{a\lambda(1-z)}} = \frac{(1-z)(1-\rho)}{1-ze^{\rho(1-z)}}.$$

8. For an M/G/1-system, the Laplace transform for the waiting time in the queue is given by

$W^*(s) = \dfrac{(1-\rho)s}{s-\lambda+\lambda B^*(s)}$, where $B^*(s)$ is the Laplace transform of the service time distribution

and $\rho = \lambda E[X]$.

(a) M/M/1-system with $E[X] = 1/\mu$:

$$B^*(s) = \frac{\mu}{\mu+s} \text{ (see, Exercise 7)}$$

$$W^*(s) = \frac{(1-\rho)s}{s-\lambda+\lambda\dfrac{\mu}{\mu+s}} = \frac{(\mu+s)(1-\rho)s}{(s-\lambda)(\mu+s)+\lambda\mu} = \frac{(s+\mu)(1-\rho)}{s+\mu-\lambda},$$ which can be further derived to $W^*(s) = (1-\rho)+\rho\cdot\dfrac{\mu-\lambda}{\mu-\lambda+s}$.

An explanation to this expression is that with probability 1-ρ, the waiting time becomes zero since the job arrives at an empty system. With probability ρ, the waiting time is exponentially distributed with mean $\dfrac{1}{\mu-\lambda}$.

(b) M/D/1-system with $E[X] = a$:

$$B^*(s) = e^{-as} \text{ (see, Exercise 7)}$$

$$W^*(s) = \frac{(1-\rho)s}{s-\lambda+\lambda e^{-as}} = \frac{(1-\rho)s}{s-\lambda(1-e^{-as})}$$

9. M/E_r/1-system

(a) In this system, the Laplace transform for the service time distribution is given by

$$B^*(s) = \left(\frac{\mu r}{\mu r+s}\right)^r \text{ (see, page 85),}$$

which means that $W^*(s) = \dfrac{(1-\rho)s}{s-\lambda+\lambda B^*(s)} = \dfrac{(1-\rho)s}{s-\lambda+\lambda\left(\dfrac{\mu r}{\mu r+s}\right)^r}$.

(b) Let r=5, $\lambda=0{,}8s^{-1}$, and $\mu=1s^{-1}$.

Alternative 1: Calculate W from $W^*(s)$

Start with the general formula for $W^*(s)$. Multiplicate both sides with $s-\lambda+\lambda B^*(s)$:

$$W^*(s)\cdot(s-\lambda+\lambda B^*(s)) = (1-\rho)s$$

Derivate both sides once:

$$\frac{d}{ds}W^*(s)\cdot(s-\lambda+\lambda B^*(s)) + W^*(s)\cdot\left(1+\lambda\frac{d}{ds}B^*(s)\right) = (1-\rho)$$

and again:

$$\frac{d^2}{ds}W^*(s)\cdot(s-\lambda+\lambda B^*(s)) + 2\cdot\frac{d}{ds}W^*(s)\left(1+\lambda\frac{d}{ds}B^*(s)\right) + W^*(s)\cdot\lambda\frac{d^2}{ds}B^*(s) = 0$$

Let $s \to 0$ ($\lim_{s \to 0} B^*(s) = 1$, $\lim_{s \to 0} W^*(s) = 1$, $\lim_{s \to 0} \frac{d}{ds} W^*(s) = -W$ etc..):

$$0 - 2W(1 - \lambda E[X]) + \lambda E[X^2] = 0$$

which means that $W = \dfrac{\lambda E[X^2]}{2(1 - \lambda E[X])}$.

For the E_r-distribution with mean $1/\mu$, $E[X^2] = \dfrac{1}{\mu^2} + \dfrac{1}{r\mu^2}$ (see, Exercise 3 in Chapter 7),

which means that $W = \dfrac{\lambda\left(\frac{1}{\mu^2} + \frac{1}{r\mu^2}\right)}{2(1 - \lambda/\mu)} = \dfrac{\rho E[X](r+1)}{2r(1-\rho)}$, where $\rho = \lambda E[X] = \lambda/\mu$.

In this specific system $W = \dfrac{0.8 \cdot 6}{10(1 - 0.8)} = 2.4$ seconds.

Alternative 2: Use the PK-formula and Little's theorem

$$\bar{N}_q = \frac{\lambda^2 E[X^2]}{2(1-\rho)} \text{ where } \rho = \lambda E[X] = \lambda/\mu$$

$$W = \frac{\bar{N}_q}{\lambda_{eff}} = \frac{\bar{N}_q}{\lambda} = \frac{\lambda E[X^2]}{2(1-\rho)}$$, which is the same formula as above...

10. There can only be 0 or 1 jobs in the system, which means that the server will have busy and idle periods as shown in the figure below:

The length of a busy period is the service time for the job currently beeing processed. An idle period starts when a job departs from the system and it ends when the next job arrives at the system.
The distribution of the busy period lengths is the service time distribution, *b(t)*, with mean *E[X]*.
Due to the Poisson process, an idle period is exponentially distributed with mean $1/\lambda$.

This means that the probability that the system is empty, $p_0 = \dfrac{1/\lambda}{E[X] + 1/\lambda}$, since it is the average fraction of time when the system is empty.
The probability that there is one job in the system is derived in the same way:

$$p_1 = 1 - p_0 = \frac{E[X]}{E[X] + 1/\lambda}$$

11. This is an M/G/1-system since the merging of two Poisson processes gives a new Poisson process. Both job streams have exponentially distributed service times.

Stream A: $\lambda_A = 1\ s^{-1}$, $E[X_A] = 1/\mu_A = 0.7$ seconds.

Stream B: $\lambda_B = 0.007\ s^{-1}$, $E[X_B] = 1/\mu_B = 30$ seconds.

(a) The waiting time distribution will be the same for both types of jobs.
The service time distribution for a random job is given by

$$b(t) = \frac{\lambda_A}{\lambda_A + \lambda_B} b_A(t) + \frac{\lambda_B}{\lambda_A + \lambda_B} b_B(t)$$

which means that the Laplace transform for the service time is

$$B^*(s) = \frac{\lambda_A}{\lambda_A + \lambda_B} B_A^*(s) + \frac{\lambda_B}{\lambda_A + \lambda_B} B_B^*(s)$$

Since both service time distributions are exponential:

$$B_A^*(s) = \frac{\mu_B}{\mu_B + s} \qquad B_B^*(s) = \frac{\mu_B}{\mu_B + s}$$

Therefore, the Laplace transform for the waiting time is $W^*(s) = \dfrac{(1-\rho)s}{s - \lambda + \lambda B^*(s)}$ where

$\lambda = \lambda_A + \lambda_B$, $\rho = \lambda E[X]$, and $E[X] = \dfrac{\lambda_A}{\lambda_A + \lambda_B} E[X_A] + \dfrac{\lambda_B}{\lambda_A + \lambda_B} E[X_B]$.

(b) The time in the system for a job is the waiting time plus the service time. This means that the Laplace transform for the time in the system for a type B job is givenby

$$S_B^*(s) = W^*(s) \cdot B_B^*(s)$$

(c) The average time in the system for a type B job, $T_B = W + E[X_B]$ where W is average waiting time in the queue.

Also, $W = \dfrac{\lambda E[X^2]}{2(1 - \lambda E[X])}$ (see, Exercise 9).

where $E[X^2] = \dfrac{\lambda_A}{\lambda_A + \lambda_B} E[X_A^2] + \dfrac{\lambda_B}{\lambda_A + \lambda_B} E[X_B^2]$.

For this system:

$$E[X] = \frac{1}{1 + 0.007} \cdot 0.7 + \frac{0.007}{1 + 0.007} \cdot 30 = 0.9037 \text{ seconds}$$

$$E[X_A^2] = 2 \cdot E[X_A]^2 = 0.98 \text{ (see, Exercise 1)}$$

$$E[X_B^2] = 2 \cdot E[X_B]^2 = 1800$$

$$E[X^2] = \frac{1}{1 + 0.007} \cdot 0.98 + \frac{0.007}{1 + 0.007} \cdot 1800 \approx 13.49 \text{ s}^2$$

$$W = \frac{\lambda E[X^2]}{2(1 - \lambda E[X])} = \frac{1.007 \cdot 13.49}{2(1 - 0.91)} \approx 75.44 \text{ seconds}$$

and $T_B = 30 + 75.44 = 105.44$ seconds

12. This is an M/G/1-system since the merging of two Poisson processes gives a new Poisson process.

Type 1 signals: $\lambda_1 = 20\ s^{-1}$, $E[X_1] = 27$ milliseconds = 0.027 seconds.

Type 2 signals: $\lambda_2 = 2\ s^{-1}$, $E[X_2] = 150$ milliseconds = 0.150 seconds.

(a) The interarrival times are exponentially distributed with mean $1/(\lambda_1 + \lambda_2)$ seconds, which means that the density function for the arrival times is given by $a(t) = (\lambda_1 + \lambda_2)e^{-(\lambda_1 + \lambda_2)t}$.

(b) The service times for a specific signal type are deterministic.

The probability that a job is of type i is $\frac{\lambda_i}{\lambda_1 + \lambda_2}$. This means that the density function for a random job is $b(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2}\delta(t - 0.027) + \frac{\lambda_2}{\lambda_1 + \lambda_2}\delta(t - 0.150)$.

(c) The mean service time for a random job is $E[X] = \frac{\lambda_1}{\lambda_1 + \lambda_2}E[X_1] + \frac{\lambda_2}{\lambda_1 + \lambda_2}E[X_2]$.

The utilisation U is given by

$$U = \lambda_{eff}E[X] = (\lambda_1 + \lambda_2)\left[\frac{\lambda_1}{\lambda_1 + \lambda_2}E[X_1] + \frac{\lambda_2}{\lambda_1 + \lambda_2}E[X_2]\right]$$

$$= \lambda_1 E[X_1] + \lambda_2 E[X_2] = 0.84$$

(d) The waiting time distribution will be the same for both types of jobs.

PK-formula plus Little's theorem gives $W = \frac{\lambda E[X^2]}{2(1 - \lambda E[X])}$ (see. Exercise 9).

where $E[X^2] = \frac{\lambda_1}{\lambda_1 + \lambda_2}E[X_1^2] + \frac{\lambda_2}{\lambda_1 + \lambda_2}E[X_2^2]$. (see. Exercise 5).

Since the service times are deterministic, $E[X_i^2] = E[X]^2$ (see. Exercise 3), which means that $E[X^2] = 0.0027$ and $W = \frac{22 \cdot 0.0027}{2(1 - 0.84)} \approx 0.186$ seconds.

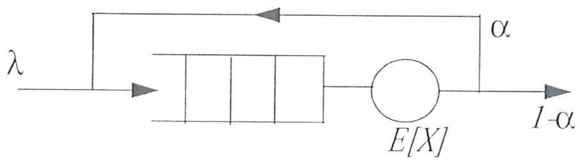
(e) Little's theorem: $\bar{N}_q = \lambda_{eff} \cdot W = \lambda W = 22 \cdot 0.186 \approx 4.1$

(f) P(a job must wait in queue)=P(system not empty)=1-p_0=$\lambda E[X] = 0.84$.

(g) A job that arrives when the system is empty will not wait in queue. This means that: $W = p_0 \cdot 0 + (1 - p_0)W_{wait}$ where W_{wait} is the waiting time for those jobs that must wait.

Therefore, $W_{wait} = \frac{W}{1 - p_0} \approx 0.22$ seconds.

13. The system is shown in the figure below



(a) Assume that a job is served n times.

The probability that a job is served n times is given by $P(n) = \alpha^{n-1}(1 - \alpha) \qquad n \geq 1$.
Assume that x_i is the i:th service time. All x_i have the same density function, with Laplace transform $B^*(s)$.

Let x_{tot} be the total service time for a job, $x_{tot} = \sum_{i=1}^{n} x_i$.

The Laplace transform for the density function for x_{tot}, given that the job is served n times, is therefore given by $B_{sys}^*(s|n) = [B^*(s)]^n$ (use the formulas for sums of random variables in Chapter 7).
Use the theorem of total probability to remove the condition on n:

$$B^*_{sys}(s) = \sum_{n=1}^{\infty} B_{sys}^*(s|n)P(n) = \sum_{n=1}^{\infty} [B^*(s)]^n \alpha^{n-1}(1-\alpha)$$

$$= B^*(s)(1-\alpha)\sum_{n=1}^{\infty} [\alpha B^*(s)]^{n-1} = B^*(s)(1-\alpha)\cdot\frac{1}{1-\alpha B^*(s)}$$

(b) **Mean service time:**

Let $\bar{x}_{tot}$ be the mean total service time for a job. $\bar{x}_{tot} = -\lim_{s\to 0}\frac{d}{ds}B^*_{sys}(s)$.

Use the equation above, multplicate both sides with $1-\alpha B^*(s)$:

$$B^*_{sys}(s)(1-\alpha B^*(s)) = B^*(s)(1-\alpha)$$

Derivate both sides:

$$\frac{d}{ds}B^*_{sys}(s)\cdot(1-\alpha B^*(s)) - \alpha B^*_{sys}(s)\frac{d}{ds}B^*(s) = (1-\alpha)\frac{d}{ds}B^*(s)$$

Let $s\to 0$ ($B^*(0) = 1$, $B^*_{sys}(0) = 1$):

$$-\bar{x}_{tot}\cdot(1-\alpha) + \alpha\bar{x} = -(1-\alpha)\bar{x}$$

Derive an expression for $\bar{x}_{tot}$: $\bar{x}_{tot} = \dfrac{\bar{x}}{1-\alpha}$

Variance of service time:

Let $\overline{x^2_{tot}}$ be the second moment of the total service time, and $\overline{x^2}$ be the second moment of one service time.
Derivate both sides of the equation above once more:

$$\frac{d^2}{ds^2}B^*_{sys}(s)\cdot(1-\alpha B^*(s)) - 2\alpha\frac{d}{ds}B^*_{sys}(s)\frac{d}{ds}B^*(s) + \alpha B^*_{sys}(s)\frac{d^2}{ds^2}B^*(s)$$

$$= (1-\alpha)\frac{d^2}{ds^2}B^*(s)$$

Let $s\to 0$:

$$\overline{x^2_{tot}}(1-\alpha) - 2\alpha\bar{x}_{tot}\bar{x} + \alpha\overline{x^2} = (1-\alpha)\overline{x^2}$$

Derive an expression for $\overline{x^2_{tot}}$ (use that $\bar{x}_{tot} = \dfrac{\bar{x}}{1-\alpha}$): $\overline{x^2_{tot}} = \dfrac{1}{1-\alpha}\left(\overline{x^2} + \dfrac{2\alpha}{1-\alpha}(\bar{x})^2\right)$.

The variance of x_{tot}, $V[X_{tot}]$, is given by

$$V[X_{tot}] = \overline{x^2_{tot}} - (\bar{x}_{tot})^2 = \frac{1}{1-\alpha}\left(\overline{x^2} + \frac{2\alpha}{1-\alpha}(\bar{x})^2\right) - \left(\frac{\bar{x}}{1-\alpha}\right)^2 = \ldots$$

$$= \frac{1}{1-\alpha}V[X] + \alpha(\bar{x}_{tot})^2$$

where $V[X]$ is the variance of a single service time.

14. Groups arrive according to a Poisson process with mean λ. Each group contain i jobs with probability g_i.

(a) Consider a time interval $[0, t]$.

Assume that n groups arrive during the interval. P(n groups arrive)=$\frac{(\lambda t)^n}{n!}e^{-\lambda t}$, $n \geq 0$.

Let m_j be the number of jobs that arrive in group j. $P(m_j = i) = g_i$. The generating function for m_j is $G(z) = \sum_{i=1}^{\infty} g_i z^i$. All m_j have the same probability distribution.

Let X be the number of *jobs* that arrive during interval $[0,t]$.

X is the sum of all m_j (if n=0, then X=0). Therefore, $X = \sum_j m_j$.

This means that the generating function for X, conditioned on n groups is given by

$P_X(z|n) = [G(z)]^n$

The theorem of total probability is used to remove the condition on n:

$$P_X(z) = \sum_{n=0}^{\infty} P_X(z|n) \cdot P(n) = \sum_{n=0}^{\infty} [G(z)]^n \frac{(\lambda t)^n}{n!} e^{-\lambda t} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t G(z))^n}{n!}$$

$$= e^{-\lambda t} \cdot e^{\lambda t G(z)} = e^{-\lambda t(1 - G(z))}$$

(b) Now we must condition on t as well.
Let T be the length of a service time.
Let ν be the number of jobs that arrive during a service time.
The density function for T is $b(t)$, with Laplace transform $B^*(s)$.

This means that $P_\nu(z|T = t) = e^{-\lambda t(1 - G(z))}$.

Use the theorem of total probability to remove the condition on T:

$$P_\nu(z) = \int_0^{\infty} P_\nu(z|T = t) b(t) dt = \int_0^{\infty} e^{-\lambda t(1 - G(z))} b(t) dt$$

This is the same as the Laplace transform for $b(t)$ evaluated in $s=\lambda(1-G(z))$,
that is $P_\nu(z) = B^*(\lambda - \lambda G(z))$