14. Groups arrive according to a Poisson process with mean λ. Each group contain i jobs with probability g_i.

(a) Consider a time interval $[0, t]$.

Assume that n groups arrive during the interval. P(n groups arrive)=$\frac{(\lambda t)^n}{n!}e^{-\lambda t}$, $n \geq 0$.

Let m_j be the number of jobs that arrive in group j. $P(m_j = i) = g_i$. The generating function for m_j is $G(z) = \sum_{i=1}^{\infty} g_i z^i$. All m_j have the same probability distribution.

Let X be the number of *jobs* that arrive during interval $[0,t]$.

X is the sum of all m_j (if n=0, then X=0). Therefore, $X = \sum_j m_j$.

This means that the generating function for X, conditioned on n groups is given by

$P_X(z|n) = [G(z)]^n$

The theorem of total probability is used to remove the condition on n:

$$P_X(z) = \sum_{n=0}^{\infty} P_X(z|n) \cdot P(n) = \sum_{n=0}^{\infty} [G(z)]^n \frac{(\lambda t)^n}{n!} e^{-\lambda t} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t G(z))^n}{n!}$$

$$= e^{-\lambda t} \cdot e^{\lambda t G(z)} = e^{-\lambda t(1 - G(z))}$$

(b) Now we must condition on t as well.
Let T be the length of a service time.
Let v be the number of jobs that arrive during a service time.

The density function for T is $b(t)$, with Laplace transform $B^*(s)$.

This means that $P_v(z|T = t) = e^{-\lambda t(1 - G(z))}$.

Use the theorem of total probability to remove the condition on T:

$$P_v(z) = \int_0^{\infty} P_v(z|T = t)b(t)dt = \int_0^{\infty} e^{-\lambda t(1 - G(z))} b(t)dt$$

This is the same as the Laplace transform for $b(t)$ evaluated in $s=\lambda(1-G(z))$, that is $P_v(z) = B^*(\lambda - \lambda G(z))$

Solutions for the exercises in Chapter 6 (Queueing networks)

1. Since system 1 is an M/M/1-system with infinite queue, the arrivals at system 2 and 3 will be Poisson distributed with means λ_2 and λ_3 jobs per time unit. Also, $\lambda_2 = \alpha \cdot \lambda_1 = 7$ and $\lambda_3 = (1-\alpha) \cdot \lambda_1 = 3$.

   (a) For a single server queue, the uilisation $U = N_s = \lambda_{eff} \cdot E[X]$.

   System 1: $U_1 = \lambda_1 \cdot \frac{1}{\mu_1} = \frac{2}{3} \approx 0,67$

   System 2: $U_2 = \lambda_2 \cdot \frac{1}{\mu_2} = \frac{7}{10} = 0,7$

   System 3: $U_3 = \lambda_3 \cdot \frac{1}{\mu_3} = \frac{3}{5} = 0,6$

   This means that system 2 has the highest utilisation.

   (b) The average total number of jobs in the system is given by $\overline{N}_{tot} = \overline{N}_1 + \overline{N}_2 + \overline{N}_3$ where $\overline{N}_m$ is the average number of jobs in system m.

   Each system is an M/M/1-systems, which means that $\overline{N}_m = \frac{\rho_m}{1-\rho_m}$ där $\rho_m = \frac{\lambda_m}{\mu_m}$.

   This gives $\overline{N}_1 = 2$, $\overline{N}_2 \approx 2,33$, $\overline{N}_3 = 1,5$ and $\overline{N}_{tot} \approx 5,83$.

   (c) Let W_{tot} be the average total waiting time for a random job.

   Little's theorem gives that $W_{tot} = \overline{N}_{qtot}/\lambda$ where $\overline{N}_{qtot}$ is the average total number of jobs in the queues.

   $\overline{N}_{qtot} = \overline{N}_{q1} + \overline{N}_{q2} + \overline{N}_{q3}$ where $\overline{N}_{qm}$ is the average number of jobs in queue m.

   Also, $\overline{N}_{qm} = \overline{N}_m - N_{s,m}$ where $N_{s,m} = \lambda_m \cdot E[X] = U_m$ is the average number of jobs in server m.

   This means that $\overline{N}_{q1} \approx 1,33$, $\overline{N}_{q2} \approx 1,63$, $\overline{N}_{q3} \approx 0,9$ and $\overline{N}_{qtot} \approx 3,86$ which gives $W_{tot} \approx 0,39$.

2. The arrival processes for system 2 and 3 are Poissonian, which means that system 2 is an M/M/1-system and system is an M/M/2 loss system (Erlang).

   (a) $\lambda_1 = \lambda = 1/15$, $\lambda_2 = \beta\lambda_1 = 1/75$, $\lambda_3 = (1-\beta)\lambda_1 = 4/75$.

   (b) Let $\rho_m = \lambda_m \cdot \bar{x}_m$, which gives $\rho_1 = 2/3$, $\rho_2 = 2/5$, and $\rho_3 = 16/15$.

   Systems 1 and 2 are M/M/1, which give $\overline{N}_1 = \frac{\rho_1}{1-\rho_1} = 2$ and $\overline{N}_2 = \frac{\rho_2}{1-\rho_2} = \frac{2}{3}$.

   System 3 is an Erlang system, which means that

   $\overline{N}_3 = \overline{N}_{s,3} = \lambda_{eff,3} \cdot \bar{x}_3 = \rho_3(1 - E_2(\rho_3)) \approx 0.8364$.

   (To find $E_2(\rho_3)$, either use the Erlang tables and intrapolate or calculate it directly).

   (c) The average blocked rate of jobs in system 3: $\lambda_{b,3} = \lambda_3 \cdot E_2(\rho_3) \approx 0.0115$ jobs per time unit.

(d) A job can choose one of three ways through the network:
A: System 1 + System 2
B: System 1 + System 3 (no blocking)
C: System 1+ Blocked at system 3
Let T_i = Total average time in the system for a job that chooses way i.

$T_A = T_1 + T_2$, $T_B = T_1 + T_3$, and $T_C = T_1$.

System 1 and 2 are M/M/1, which means that $T_1 = \dfrac{\bar{x}_1}{1-\rho_1} = 30$ and $T_2 = \dfrac{\bar{x}_2}{1-\rho_2} = 50$.

System 3 is a loss system, which means that $T_3 = \bar{x}_3 = 20$.

Therefore, $T_A = 80$, $T_B = 50$, and $T_C = 30$.

A job that is not blocked has gone way A or B. We need to find the probabilities that a job that is not blocked goes way A and B respectively.
Let Λ_i = average number of jobs per time unit that chooses way i.

This means that $\Lambda_A = \lambda_2 = 1/75$, $\Lambda_B = \lambda_3(1-E_2(\rho_3)) \approx 0.0427$, and
$\Lambda_C = \lambda_3 E_2(\rho_3) \approx 0.0107$.

Note that $\Lambda_A + \Lambda_B + \Lambda_C = \lambda$ since a job must choose one of the ways.

Now,

P(a job chooses way A (of A and B))=$p_A = \dfrac{\Lambda_A}{\Lambda_A + \Lambda_B} \approx 0.2375$

P(a job chooses way B (of A and B))=$p_B = \dfrac{\Lambda_B}{\Lambda_A + \Lambda_B} \approx 0.7625$

Let T_{tot} = average total time in the system for a job that is not blocked.

The theorem of total probability gives that $T_{tot} = p_A \cdot T_A + p_B \cdot T_B \approx 57$ time units.

(e) Do the same steps as in (d), but use W instead.

$W_1 = T_1 - \bar{x}_1 = 20$, $W_2 = T_2 - \bar{x}_2 = 20$, $W_3 = 0$.

$W_A = W_1 + W_2 = 40$, $W_B = W_1 + W_3 = 20$, $W_c = W_1 = 20$.

$W_{tot} = p_A \cdot W_A + p_B \cdot W_B \approx 25$.

3. The arrival process at system 2 is Poissonian.

(a) Arrival rate: $\lambda_2 = (1-\alpha)\lambda = 0.04$ jobs per second.

Offered load: $\rho_2 = \lambda_2 \cdot \bar{x}_2 = 2.4$ Erlangs

(b) P(a job is blocked at system 2)=$E_3(\rho_2) \approx 0.2684$.

(c) System 1 is an M/M/1, which gives $\bar{N}_1 = \dfrac{\lambda_1 \bar{x}_1}{1-\lambda_1 \bar{x}_1} = 1$.

System 2 is an M/M/3 loss system, which means that $\bar{N}_2 = \bar{N}_{s,2} = \rho_2(1-E_3(\rho_2)) \approx 1.8$.

(d) Carried load: $\rho_{c,2} = \bar{N}_2 \approx 1.8$ Erlangs.

Blocked load: $\rho_{b,2} = \rho_2 - \rho_{c,2} = 0.6$ Erlangs.

(e) $\lambda_{ut,2} = \lambda_{eff,2} = \lambda_2(1-E_3(\rho_3)) \approx 0.029$ jobs per second.

4. Let λ_k=average arrival rate for system k. λ_1=4 and λ_2=2 are given in the text.
From the figure we see that $\lambda_3 = \lambda_1 + \lambda_2 = 6$, $\lambda_4 = \alpha\lambda_3 = 4$ and $\lambda_5 = (1-\alpha)\lambda_3 = 2$.
Also, let $\rho_k = \lambda_k/\mu_k$ which means that $\rho_1 = 0,8$, $\rho_2 = 0,4$, $\rho_3 = 0,75$, $\rho_4 = 2$ and $\rho_5 = 1/3$.

(a) Let $\overline{N}_k$ the average number of jobs in system k.

$$\overline{N}_1 = \frac{\rho_1}{1-\rho_1} = 4,\ \overline{N}_2 = \frac{\rho_2}{1-\rho_2} = \frac{2}{3},\ \overline{N}_3 = \frac{\rho_3}{1-\rho_3} = 3,$$

$$\overline{N}_4 = \rho_4 \cdot (1 - E_3(\rho_4)) \approx 1.58 \text{ and } \overline{N}_5 = \frac{\rho_5}{1-\rho_5} = \frac{1}{2}$$

(b) Jobs can only be blocked in system 4.
Average rate of blocked jobs = $\lambda_b = \lambda_4 \cdot E_3(\rho_4) \approx 0,84$.

(c) Let T_k be the average time in system k. Little's theorem gives:

$$T_1 = \frac{\overline{N}_1}{\lambda_1} = 1,\ T_2 = \frac{\overline{N}_2}{\lambda_2} \approx 0.333,\ T_3 = \frac{\overline{N}_3}{\lambda_3} = 0.5,\ T_4 = \frac{1}{\mu_4} = 0.5 \text{ and}$$

$$T_5 = \frac{\overline{N}_5}{\lambda_5} = 0.25$$

A job *that is not blocked* must have chosen one of the following four paths:
Path A: System 1+3+4; Path B: 1+3+5; Path C: 2+3+4; Path D: 2+3+5
Let T_i be the total average time in the system for a job that chooses path i and that is not blocked.

$$T_A = T_1 + T_3 + T_4 = 2,\ T_B = T_1 + T_3 + T_5 = 1.75,$$

$$T_C = T_2 + T_3 + T_4 \approx 1.333 \text{ and } T_D = T_1 + T_3 + T_5 \approx 1.083$$

Let Λ_i = average rate of jobs that chooses path i and that is not blocked.

$$\Lambda_A = \lambda_1\alpha(1 - E_3(\rho_4)) \approx 2.105,\ \Lambda_B = \lambda_1(1-\alpha) \approx 1.333,$$

$$\Lambda_C = \lambda_2\alpha(1 - E_3(\rho_4)) \approx 1.053 \text{ and } \Lambda_D = \lambda_2(1-\alpha) \approx 0.667$$

This means that P(a job chooses path i)=$p_i = \dfrac{\Lambda_i}{\Lambda_A + \Lambda_B + \Lambda_C + \Lambda_D}$ and therefore

$p_A \approx 0,408$, $p_B \approx 0,258$, $p_C \approx 0,204$ and $p_D \approx 0,129$

The theorem of total probability gives that the average total time in the system for a random job that is not blocked, $T_{tot} = \sum_{i=A}^{D} p_i \cdot T_i \approx 1.7$ seconds.

(d) Let W = total average waiting time for a random job (also jobs that are blocked).
Little's theorem gives that $W = \overline{N}_{qtot}/\lambda_{eff}$ where $\overline{N}_{qtot}$ is the total average number of jobs in the queues and λ_{eff} is the average rate of jobs that enters the network, that is $\lambda_{eff} = \lambda_1 + \lambda_2 = 6$.

$\overline{N}_{qtot} = \sum_{k=1}^{5} \overline{N}_{qk}$ where $\overline{N}_{qk}$ is the average number of jobs in queue k.

p_0 is found by using the normalisation condition: $\sum_{k=0}^{5} p_k = 1 \Rightarrow p_0 = \frac{5}{24}$.

The average number of jobs in system 1, $\bar{N}_1 = 1 \cdot p_{1,0} + 1 \cdot p_{1,1} + 1 \cdot p_{1,2} = \frac{1}{2}$.

The average number of jobs in system 1,

$\bar{N}_2 = 1 \cdot p_{0,1} + 1 \cdot p_{1,1} + 2 \cdot p_{0,2} + 2 \cdot p_{1,2} = \frac{7}{12}$

(b) Average rate of jobs that are served in system2.

Alternative 1: $\lambda_{eff,2} = \mu p_{1,0} + \mu p_{1,1} = \frac{11}{24}$

Alternative 2: $\lambda_{eff,2} = \lambda_{ut,2} = \mu p_{0,1} + \mu p_{1,1} + 2\mu p_{0,2} + 2\mu p_{1,2} = \frac{11}{24}$.

Little's theorem gives that $T_2 = \frac{\bar{N}_2}{\lambda_{eff,2}} = \frac{14}{11}$.

(c) If the systems are independent is the following condition fulfilled:
P(k jobs in system 1 and m jobs in system 2)=P(k jobs in system 1)P(m jobs in system 2).
Example:

P(0 jobs in system 1 and 0 jobs in system 2)= $p_{0,0} = \frac{5}{24}$.

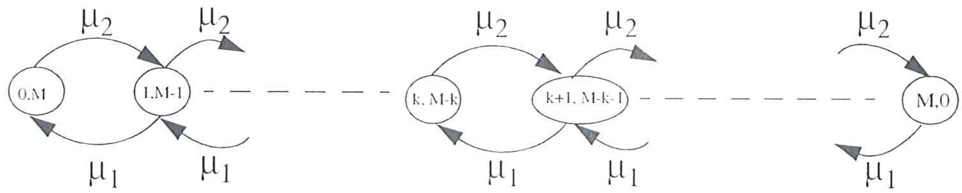
P(0 jobs in system 1)=$p_{0,0} + p_{0,1} + p_{0,2} = \frac{1}{2}$

P(0 jobs in system 2)=$p_{0,0} + p_{1,0} = \frac{13}{24}$

This means that the condition is not fulfilled since $\frac{1}{2} \cdot \frac{13}{24} \neq \frac{5}{24}$.

7.

(a) Let state i,j = i jobs in system 1 and j jobs in system 2.
$p_{i,j}$=P(i jobs in system 1 and j jobs in system 2)
State diagram:



The "cut-method" gives the following global balance equations:

$$\mu_2 p_{k,M-k} = \mu_1 p_{k+1,M-k-1} \qquad 0 \le k < M$$

which means that $p_{k,M-k} = \left(\frac{\mu_2}{\mu_1}\right)^k p_{0,M}$.

$p_{0,M}$ is found with the normalisation condition:

$$\sum_{k=0}^{M} p_{k,M-k} = 1 \Rightarrow p_{0,M} = \frac{1 - \mu_2/\mu_1}{1 - (\mu_2/\mu_1)^{M+1}}$$

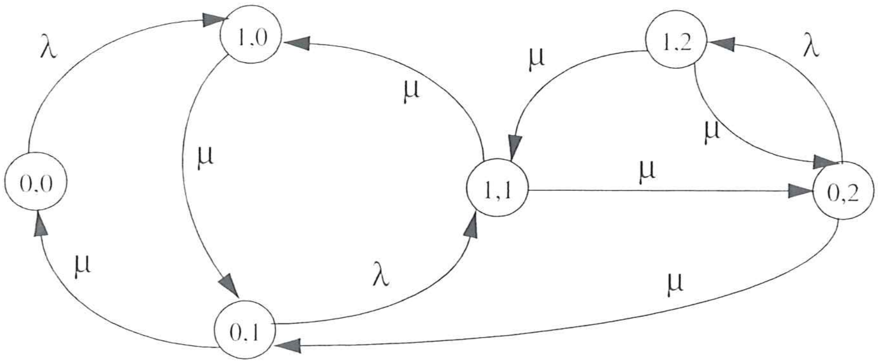
$$\overline{N}_{q1} = \frac{\rho_1^2}{1-\rho_1} = 3.2\,,\ \overline{N}_{q2} = \frac{\rho_2^2}{1-\rho_2} \approx 0.267\,,\ \overline{N}_{q3} = \frac{\rho_3^2}{1-\rho_3} = 2.25\,,$$

$$\overline{N}_{q4} = 0 \text{ and } \overline{N}_{q5} = \frac{\rho_5^2}{1-\rho_5} \approx 0.167 \text{ which means that } \overline{N}_{qtot} \approx 5.88\,.$$

Therefore, $W \approx 0.98$ seconds.

5. Note that the arrival process for system 2 *is not* Poissonian!
   Offered load for system 1=Offered load for the whole system: $\rho = \lambda \cdot \bar{x}$.
   Offered load for system 2 = Blocked load for system 1: $\rho_{o,2} = \rho_{b,1} = \rho E_{m_1}(\rho)$.
   Blocked load for system 2 = Blocked load for the whole system: $\rho_{b,2} = \rho_b = \rho E_{m_1+m_2}(\rho)$.
   Explanation: A job is blocked when all servers are busy. For this job, the system acts as an M/M/(m_1+m_2) loss system with offered load ρ.
   Carried load for system 2: $\rho_{c,2} = \rho_{o,2} - \rho_{b,2} = \rho E_{m_1}(\rho) - \rho E_{m_1+m_2}(\rho)$.

6. This system must be analysed with a Markov chain.
   State i,j: i jobs in system 1, j jobs in system 2.
   A job that departs from system 1 becomes an arrival at system 2.
   λ=1=arrival intensity, μ=1=service intensity.
   State diagram:



   (a) Let $p_{0,0} = p_0, p_{1,0} = p_1, p_{0,1} = p_2, p_{1,1} = p_3, p_{0,2} = p_4$, and $p_{1,2} = p_5$
   "Flow-in-flow-out" method gives the following global balance equations:

$$\left.\begin{aligned} \lambda p_0 &= \mu p_1 \\ \mu p_1 &= \lambda p_0 + \mu p_3 \\ (\lambda+\mu)p_2 &= \mu p_1 + \mu p_4 \\ 2\mu p_3 &= \mu p_5 + \lambda p_2 \\ (\lambda+\mu)p_4 &= \mu p_5 + \mu p_3 \\ 2\mu p_5 &= \lambda p_4 \end{aligned}\right\} \Rightarrow [\lambda = \mu = 1] \Rightarrow \begin{cases} p_1 = \frac{8}{5}p_0 \\ p_2 = p_0 \\ p_3 = \frac{3}{5}p_0 \\ p_4 = \frac{2}{5}p_0 \\ p_5 = \frac{1}{5}p_0 \end{cases}$$

(b) Let U_1= utilisation of server 1, U_1=P(server 1 is busy):

$$U_1 = \sum_{k=1}^{M} p_{k,M-k} = 1 - p_{0,M} = \ldots = (\mu_2/\mu_1) \cdot \frac{1-(\mu_2/\mu_1)^M}{1-(\mu_2/\mu_1)^{M+1}}$$

(c) Let Λ_2 = rate of jobs that leave system 2.

$$\Lambda_2 = \mu_2 \cdot P(\text{server 2 busy}) = \mu_2 \cdot (1 - p_{M,0})$$

(d) Let U_2=utilisation of server 2.

$$U_2 = \sum_{k=0}^{M-1} p_{k,M-k} = 1 - p_{M,0} = \ldots = \frac{1-(\mu_2/\mu_1)^M}{1-(\mu_2/\mu_1)^{M+1}} = (\mu_1/\mu_2) \cdot U_1$$

This also means that $\Lambda_2 = \mu_2 \cdot U_2 = \mu_1 \cdot U_1$.

M	U_1	U_2	Λ_2
1	0.09090	0.9090	0.0909
2	0.0991	0.9910	0.0991
3	0.0999	0.9990	0.0999
5	0.0999	0.9999	0.0999

As can be seen in the table, system 2 quickly becomes a bottleneck.

8. Since new requests arrive according to a Poisson process, and systems are M/M/1 with unlimited queues, we can analyse each system independently.

(a) The following is valid:

$\lambda_1 = \lambda + \lambda_2 + \lambda_3$; $\lambda_2 = (1-\alpha)\beta\lambda_1$ and $\lambda_3 = (1-\alpha)(1-\beta)\lambda_1$.

This means that $\lambda_1 = \lambda/\alpha$, $\lambda_2 = \frac{1-\alpha}{\alpha} \cdot \beta\lambda$ and $\lambda_3 = \frac{1-\alpha}{\alpha} \cdot (1-\beta)\lambda$

(b) Let $\rho_k = \lambda_k/\mu_k$. The average total number of requests in the site, $\overline{N}_{tot} = \overline{N}_1 + \overline{N}_2 + \overline{N}_3$ where $\overline{N}_k$ is the average number of requests in system k.

Since each system is an M/M/1-system, $\overline{N}_k = \frac{\rho_k}{1-\rho_k}$, which means that

$$\overline{N}_{tot} = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} + \frac{\rho_3}{1-\rho_3}.$$

(c) Let T = the average total time a request spends in the site before completed.

Little's theorem gives that $T = \overline{N}_{tot}/\lambda_{eff}$ where λ_{eff} is the average rate of requests that enters the site. Since no requests are blocked, $\lambda_{eff} = \lambda$, which means that $T = \overline{N}_{tot}/\lambda$.

(d) Let W = average total waiting time for a request.

Little's theorem again gives $W = \overline{N}_{qtot}/\lambda$ where $\overline{N}_{qtot}$ is the average total number of requests in the queues.

$\overline{N}_{qtot} = \overline{N}_{q1} + \overline{N}_{q2} + \overline{N}_{q3}$ where $\overline{Nq_k}$ is the average number of requests in queue k.

$\overline{N}_{qk} = \frac{\rho_k^2}{1-\rho_k}$, which means that $\overline{N}_{qtot} = \frac{\rho_1^2}{1-\rho_1} + \frac{\rho_2^2}{1-\rho_2} + \frac{\rho_3^2}{1-\rho_3}$.

(e) The average number of completed requests per time unit = $\lambda_{out}=\lambda_{eff}=\lambda$.

9. Let λ_i = arrival rate for system i.

$\lambda_1 = \lambda = 10$, $\lambda_2 = \lambda + \alpha\lambda_2 \Rightarrow \lambda_2 = 25$, $\lambda_3 = \beta\lambda = 7.5$, and $\lambda_4 = (1-\beta)\lambda = 2.5$.

Let $\rho_i = \lambda_i/\mu_i$: $\rho_1 = 1/2$, $\rho_2 = 5/8$, $\rho_3 = 2.5$, and $\rho_4 = 2.5$.

(a) Let $\overline{N}_i$ = average number of jobs in system i.

$$\overline{N}_1 = \frac{\rho_1}{1-\rho_1} = 1,\ \overline{N}_2 = \frac{\rho_2}{1-\rho_2} = \frac{5}{3},$$

$$\overline{N}_3 = \rho_3 \cdot (1 - E_{10}(\rho_3)) \approx 2.5,\ \overline{N}_4 = \rho_4 \cdot (1 - E_5(\rho_4)) \approx 2.33.$$

(b) Jobs are blocked in systems 3 and 4.

$\lambda_{b,3} = \lambda_3 \cdot E_{10}(\rho_3) \approx 0.002$ jobs per time unit.

$\lambda_{b,4} = \lambda_4 \cdot E_5(\rho_4) \approx 0.174$ jobs per time unit.

$\lambda_{b,tot} = \lambda_{b,3} + \lambda_{b,4} \approx 0.18$ jobs per time unit.

(c) Let T_{tot} = total average time in the network for a job.

Little's theorem gives $T_{tot} = \overline{N}_{tot}/\lambda$ where $\overline{N}_{tot} = \overline{N}_1 + \overline{N}_2 + \overline{N}_3 + \overline{N}_4 \approx 7.497$, which means that $T_{tot} \approx 0.75$ time units.

(d) A job that is not blocked chooses one of two paths:

A: System 1+2+3

B: System 1+2+4

Let T_i = average time in system i.

Little's theorem gives $T_1 = \overline{N}_1/\lambda_1 = 0.1$ and $T_2 = \overline{N}_2/\lambda = 0.167$.

Systems 3 and 4 are loss systems, which means that

$T_3 = 1/\mu_3 \approx 0.333$ and $T_4 = 1/\mu_4 = 1$

Let T_k = total time in the network for a job that chooses path k (and is not blocked).

$T_A = T_1 + T_2 + T_3 \approx 0.600$ time units, $T_B = T_1 + T_2 + T_4 \approx 1.27$ time units

Let λ_k = average rate of jobs that chooses path k (and is not blocked)

$\lambda_A = \beta\lambda \cdot (1 - E_{10}(\rho_3)) \approx 7.5$, $\lambda_B = (1-\beta)\lambda \cdot (1 - E_5(\rho_4)) \approx 2.33$

P(a job chooses path k and is not blocked)=$\frac{\lambda_k}{\lambda_A + \lambda_B}$

The theorem of total probability gives that

$$T = \frac{\lambda_A}{\lambda_A + \lambda_B} \cdot T_A + \frac{\lambda_B}{\lambda_A + \lambda_B} \cdot T_B \approx 0.76 \text{ time units.}$$

10.Missing in the book: Let α=0.4, λ_{in}=0.75, μ_1=1.5, μ_2=2, and μ_3=3.

(a) The following equations can be derived from the figure: $\lambda_1 = \lambda_{in} + \lambda_2$, $\lambda_2 = \alpha\lambda_1$ and $\lambda_3 = (1-\alpha)\cdot\lambda_1 = \lambda_{in}$.

This means that $\lambda_1 = 1.25$, $\lambda_2 = 0.5$ and $\lambda_3 = 0.75$.

Also, $\lambda_{block} = \lambda_3 \cdot E_2(\lambda_3/\mu_3) = 0.75 \cdot E_2(0, 25) \approx 0.018$ and

$\lambda_{out} = \lambda_3 \cdot (1 - E_2(\lambda_3/\mu_3)) \approx 0.73$

(b) Let $\rho_i = \lambda_i/\mu_i$ and let $\overline{N}_i$ = average number of jobs in system i.

Then, $\overline{N}_1 = \dfrac{\rho_1}{1-\rho_1} = 5$, $\overline{N}_2 = \dfrac{\rho_2}{1-\rho_2} \approx 0.333$ (M/M/1-systems)

and $\overline{N}_3 = \rho_3 \cdot (1 - E_2(\rho_3)) \approx 0.24$ (Carried load in an Erlang system)

This means that the average total number of jobs in the network,

$\overline{N}_{tot} = \overline{N}_1 + \overline{N}_2 + \overline{N}_3 \approx 5.573$.

(c) Let T=total average time in the network for a job that is not blocked

Alternative 1:

Let T_i = average time in system i.

Little's theorem gives that: $T_1 = \dfrac{\overline{N}_1}{\lambda_1} = 4$, $T_2 = \dfrac{\overline{N}_2}{\lambda_2} \approx 0.667$ and $T_3 = \dfrac{1}{\mu_3} \approx 0.333$.

A job that is not blocked is first served one or more times in systems 1 and 2. Assume that the job is served M times in system 2.Then, the job will be served once in system 3. If M=k, the total average time in the network will be:

$E[T|M = k] = (k+1)T_1 + kT_2 + T_3$

Also, $P(M = k) = \alpha^k(1-\alpha)$.

The theorem of total probability gives:

$$E[T] = \sum_{k=0}^{\infty} E[T|M = k] \cdot P(M = k) = T_1 + T_3 + \frac{\alpha}{1-\alpha}(T_1 + T_2) \approx 7.4$$

Alternative 2:

$E[T] = T_{12} + T_3$ where T_{12} is the total average time a job spends in systems 1 och 2.

Let $\overline{N}_{12} = \overline{N}_1 + \overline{N}_2 \approx 5.333$. Little's theorem gives $T_{12} = \dfrac{\overline{N}_{12}}{\lambda_{in}} \approx 7.111$.

Also, $T_3 = \dfrac{1}{\mu_3}$. This means that $E[T] = T_{12} + T_3 \approx 7.4$.

11.In the exercises below $\rho_k = \lambda_k / \mu_k$

(a) The following equations can be derived from the figure:

$$\begin{cases} \lambda_A = \lambda + \lambda_D \\ \lambda_D = \alpha\lambda_A \\ \lambda_B = (1-\beta)\lambda \\ \lambda_C = (1-\delta)\lambda_B \end{cases}$$

This means that $\lambda_A = 50$, $\lambda_B = 1$, $\lambda_C = 0,3$ and $\lambda_D = 40$.

(b) lSystem D is an M/M/∞-system.

Let $\overline{N}_D$= average number of customers in system D.

Alternative 1:

Since there is no queue, $\overline{N}_D$ = average number of customers in the servers = Carried load in the system = Offered load, since there is no blocking.

This means that $\overline{N}_D = \lambda_D \cdot \frac{1}{\mu_D} = 400$.

Alternative 2:

Draw the state diagram and find p_k. Then, $\overline{N}_D = \sum_{k=1}^{\infty} k \cdot p_k$.

(c) Let x_{tot}= total average service time for a customer.

A customer is first served a number of times in system A.

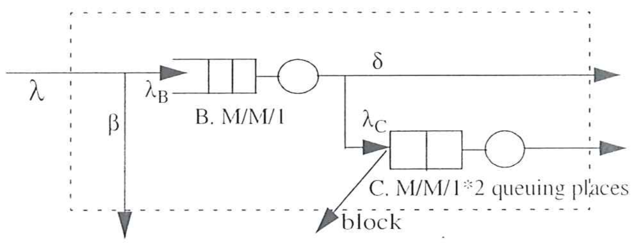
P(k services in system A)=$\alpha^{k-1}(1-\alpha)$

E[number of services in system A]=$\sum_{k=1}^{\infty} k \cdot \alpha^{k-1}(1-\alpha) = \frac{1}{1-\alpha}$

Let x_A=Total average service time in system A for a customer.

Then $x_A = \frac{1}{1-\alpha} \cdot \frac{1}{\mu_A} \approx 0.0833$ seconds.

The customer proceeds to systems B and C. Consider this as a separate network:



Apply Little's theorem on *all* customers, that is also those customers that chooses to leave the site before system B.

Let x_{BC} be the total average service time for a customer in this network.

Little's theorem gives that $x_{BC} = \overline{N}_{s,BC}/\lambda$ where $\overline{N}_{s,BC}$ is the total average number of customers in servers B and C. Also, $\overline{N}_{s,BC} = \overline{N}_{s,B} + \overline{N}_{s,C}$ where $\overline{N}_{s,k}$ is the average number of customers in server k.

$\overline{N}_{s,B} = \rho_B = 0.2$

System C is an M/M/1-system with 2 queueing places. Determine the stationary probability distribution from the state diagram.

Result: $p_0 = \frac{1}{40}$, $p_1 = \frac{3}{40}$, $p_2 = \frac{9}{40}$ and $p_3 = \frac{27}{40}$

$$\bar{N}_{s,C} = 0 \cdot p_0 + 1 \cdot (p_1 + p_2 + p_3) = 0.975$$

This means that $x_{BC} = \frac{\bar{N}_{s,B} + \bar{N}_{s,C}}{\lambda} = 0.1175$ seconds.

The total average service time for a customer is $x_{tot} = x_A + x_{BC} \approx 0.20$ seconds.

12.

(a) $\begin{cases} \lambda_i = \frac{1}{M} \cdot \left(\lambda + \alpha \cdot \sum_{i=1}^{M} \lambda_i\right) \\ \sum_{i=1}^{M} \lambda_i(1-\alpha) = \lambda \end{cases}$ which means that $\lambda_i = \frac{1}{M(1-\alpha)} \cdot \lambda$ for $i=1..M$

(b) Let $\rho_i = \frac{\lambda_i}{\mu} = \frac{\lambda}{\mu M(1-\alpha)}$ for $i=1..M$.

The average number of jobs in system i is $\bar{N}_i = \frac{\rho_i}{1-\rho_i} = \frac{\lambda}{\mu M(1-\alpha) - \lambda}$ for $i=1..M$.

This means that the total average number of jobs in the network,

$$\bar{N}_{tot} = \sum_{i=1}^{M} \bar{N}_i = \frac{M\lambda}{\mu M(1-\alpha) - \lambda}$$

Let T =total average time in the network for a job.

Little's theorem gives that $T = \bar{N}_{tot}/\lambda$ which means that $T = \frac{M}{\mu M(1-\alpha) - \lambda}$.

(c) Assume that a job is served B times. Then, $P(B = k) = \alpha^{k-1} \cdot (1-\alpha)$.
None of these services can be in system 1.

P(a job chooses not system 1)= $1 - \frac{1}{M} = \frac{M-1}{M}$ which means that

P(a job is never served in system $1|B = k) = \left(\frac{M-1}{M}\right)^k$.

The theorem of total probability gives that P(a job is never served in system 1)=

$$\sum_{k=1}^{\infty} \left(\frac{M-1}{M}\right)^k \cdot \alpha^{k-1} \cdot (1-\alpha) = \frac{(M-1)(1-\alpha)}{M - \alpha(M-1)}$$