KTH Teknikvetenskap

SF1624 Algebra och geometri
Exam
Thursday, March 17, 2016

Time: 08:00am–1:00pm
No books/notes/calculators etc. allowed
Examiner: Tilman Bauer

This exam consists of nine problems, each worth 4 points.

Part A comprises the first three problems. The bonus points from the seminars will be automatically added to the total score of this part, which however cannot exceed 12 points.

The next three problems constitute part B, and the last three problems part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

To get full score on a problem, your solution must be well-presented and easy to follow. In particular, you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument. Solutions severely lacking in these respects will achieve at most 2 points.

p.t.o.

Part A

1. The line L_1 is given by
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix},$$
and the line L_2 by
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$
   (a) Determine in parametric form the plane Π which is parallel with the line L_1 and contains the line L_2. **(2 p)**
   (b) Compute the distance between L_1 and L_2. **(2 p)**

2. Given the matrix
$$A = \begin{bmatrix} 5 & 6 & 0 \\ -3 & -4 & 0 \\ 3 & 3 & -1 \end{bmatrix}.$$
   (a) Determine all eigenvectors with respect to the eigenvalues -1 and 2. **(3 p)**
   (b) Why is the matrix A diagonalizable? **(1 p)**

3. The quadratic form Q on $\mathbb{R}^2$ is given by
$$Q(\vec{x}) = x_1^2 + x_1x_2 + x_2^2.$$
   (a) Find the symmetric matrix A which satisfies $Q(\vec{x}) = \vec{x}^T A\vec{x}$. **(1 p)**
   (b) Determine whether Q is positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite. **(3 p)**

PART B

4. Let $T_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which has standard matrix
$$A = \begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix}.$$
   (a) Let L be the line given by $2x - 3y = -11$. Show that T_A maps L onto a line $T_A(L)$. **(2 p)**
   (b) Find a line L' such that $T_A(L')$ is a point. Give an equation for L'. **(2 p)**

5. In $\mathbb{R}^4$, the following four vectors are given:
$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -3 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \quad \text{and} \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
   The vector space V is $V = \mathrm{Span}(\vec{u}, \vec{v}, \vec{w})$.
   (a) Show that $\beta = \{\vec{u}, \vec{v}\}$ is an orthogonal basis for V. **(1 p)**
   (b) We have the basis $\gamma = \{\vec{v}, \vec{w}\}$ for V. Determine the coordinate vector of $\mathrm{Proj}_V(\vec{x})$ in the basis γ. **(3 p)**

6. Let A be a symmetric 3×3-matrix. Suppose that its characteristic polynomial has a simple root $\lambda_1 = 2$ corresponding to an eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and a double root $\lambda_2 = -2$.
   (a) Let $\vec{w} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$. Compute $A^5\vec{w}$ **(2 p)**
   (b) Compute the matrix A. **(2 p)**

PART C

7. The planes P_1 and P_2 in $\mathbb{R}^3$ are given by the equations:
$$P_1: \ x - y + z = 5 \qquad P_2: \ 2x + 2z = -8$$
The line L is $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$, t an arbitrary number. The line L is reflected through P_1 onto a line L'. Determine whether L' intersects the plane P_2. **(4 p)**

8. Let
$$\beta = \{\cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(10x), \sin(10x)\}.$$
The set β forms a basis for a subspace V of the vector space of real-valued functions of one variable x. The derivative map $D\colon V \longrightarrow V$ is the linear map that sends a vector $f(x)$ in V to
$$D(f(x)) = \frac{df}{dx},$$
its derivative.
   (a) Find a matrix representation of D in the basis β. **(2 p)**
   (b) Determine whether the map D is diagonalizable. **(2 p)**

9. Let A and P be 3×3-matrices, with P invertible.
   (a) Show that $\operatorname{tr}(A) = \operatorname{tr}(P^{-1}AP)$, where tr denotes the trace of a matrix. **(2 p)**
   (b) Suppose that A is diagonalizable and satisfies the conditions
$$\begin{aligned} \operatorname{tr}(A) &= 0, \\ \operatorname{tr}(A^2) &= 14, \\ \operatorname{tr}(A^3) &= -18. \end{aligned}$$
   Compute $\det(A)$. **(2 p)**